ONE-DIMENSIONAL ADIABATIC CHILD-LANGMUIR FLOW

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Abstract

A theory is presented that describes steady-state one-dimensional Child-Langmuir flow at a self-consistent finite temperature distribution. In particular, warm-fluid equations and adiabatic equation of state are used to derive the self-consistent Poisson equation. The profiles of the charged-particle density, the velocity, the electrostatic potential, the pressure and the temperature are computed. The adiabatic equation of state assures the conservation of normalized (longitudinal) rms thermal emittance of any small segment of the flow. Good agreement is found between the approximation of the adiabatic equation of state and self-consistent simulation.

INTRODUCTION

Recently, there are renewed interests in the research and development of high-brightness dc thermionic electron or ion guns. The renewed interests arise from a) applications of high-brightness dc electron guns in x-ray free electron laser applications [1], and b) utilizations of high-brightness dc electron guns in the realization of adiabatic thermal beam equilibria in periodic focusing fields [2-5].

The 1D cold-fluid Child-Langmuir (C-L) flow [6,7] is an important aspect of dc thermionic gun theory. It corresponds to the cold-fluid equilibrium of a charged-particle flow between two plates with an electrostatic potential bias.

The 1D adiabatic thermal C-L flow discussed in this paper corresponds to a warm-fluid equilibrium of a charged-particle flow between two plates with an electrostatic potential bias under the adiabatic equation of state. A warm-fluid description of the 1D adiabatic thermal C-L flow is presented. A self-consistent Poisson equation is derived. In the limit of zero temperature, the self-consistent Poisson equation recovers the corresponding Poisson equation for the cold-fluid C-L flow. An effort is initiated to validate the theory via self-consistent simulation. To date, good agreement has been found between the approximation of the adiabatic equation of state and self-consistent simulation in the high-temperature regime.

THEORY OF ONE-DIMENSIONAL ADIABATIC THERMAL CHILD-LANGMUIR FLOW

We consider the non-relativistic 1D C-L flow under the influence of a finite temperature profile between two conducting plates located at $z=0$ and $z=d$. The adiabatic warm-fluid equations in cgs units are:

$$mnV\frac{\partial V}{\partial z} = -qn\frac{\partial \phi}{\partial z} - \frac{\partial p}{\partial z}, \quad (1)$$

$$\frac{\partial nV}{\partial z} = 0, \quad (2)$$

$$\partial^2\phi/\partial z^2 = -4\pi qn, \quad (3)$$

$$\frac{\partial}{\partial z}\left(\frac{p}{n^3}\right) = 0, \quad (4)$$

$$p = k_B nT, \quad (5)$$

where n, V, ϕ, p and T are the equilibrium density, flow velocity, electrostatic potential, longitudinal pressure and temperature profiles, respectively, k_B is the Boltzmann constant, and m and q are the rest mass and charge of the charged particle, respectively.

Equation (4) is the one-dimensional adiabatic equation of state, which can be derived in [8]. It is a statement of entropy conservation. For any small segment of the flow, the normalized rms thermal emittance $\varepsilon_n = \gamma\beta\varepsilon$ is proportional to p/n^3 or T/n^2, where ε is the unnormalized rms thermal emittance, $\gamma = (1-V^2/c^2)^{-1/2} \cong 1$ is the relativistic mass factor, and $\beta = V/c$ with c being the speed of light in vacuum. Therefore, the adiabatic equation of state assures that the normalized rms thermal emittance of any small segment of the flow is conserved.

Substituting Eqs. (4) and (5) into Eq. (1) yields

$$\frac{\partial}{\partial z}\left(\frac{1}{2}mV^2 + q\phi + \frac{3p}{2n}\right) = 0, \quad (6)$$

$$\frac{\partial}{\partial z}\left(\frac{1}{2}mV^2 + q\phi + \frac{3}{2}k_B T\right) = 0. \quad (7)$$

Equation (6) or (7) is an important conservation law in addition to charge conservation, i.e., $J = qnV = const.$ and entropy conservation, i.e., $p/n^3 = const.$

Because

$$T/n^2 = const., \quad (8)$$

from Eqs. (4) and (5), we express Eq. (7) as

$$\frac{1}{2}mV^2 + q\phi + \frac{3k_B}{2q^2}\frac{T}{n^2}\frac{J^2}{V^2} = C, \quad (9)$$

where the constant C is defined by

$$C = \frac{1}{2}mV^2(0) + \frac{3}{2}k_B T_c, \quad (10)$$

T_c is the emitter temperature, and use has been made of the boundary condition $\phi(0)=0$. The solutions to Eq. (9) are

$$V^2 = \left(V^2\right)_\pm = -\frac{(q\phi - C)}{m} \pm \left[\frac{(q\phi - C)^2}{m^2} - \frac{3J^2 k_B T}{q^2 mn^2}\right]^{1/2}. \quad (11)$$

Because $q\phi < 0$ in the gun, the solution of interest is

$$V^2 = \left(V^2\right)_+ = -\frac{(q\phi - C)}{m} + \left[\frac{(q\phi-C)^2}{m^2} - \frac{3J^2 k_B T}{q^2 m n^2}\right]^{1/2} \quad (12)$$

which, in the cold limit ($T = 0$), gives

$$V = V_{cold} \equiv \left(\frac{-2q\phi}{m}\right)^{1/2} . \quad (13)$$

In order for V^2 to be real, we must have

$$q\phi - C \le -\left[\frac{3mJ^2}{q^2}\left(\frac{k_B T}{n^2}\right)\right]^{1/2} . \quad (14)$$

Using the boundary conditions at the emitter, i.e., $\phi(z=0)=0$, $T(z=0)=T_c$ and $n(z=0)=n(0)$, we find that the critical value of C at which the equal sign in Eq. (14) holds is

$$C = C_{crit} \equiv \left\{\frac{3mJ^2}{q^2}\left[\frac{k_B T_c}{n^2(0)}\right]\right\}^{1/2} . \quad (15)$$

Substituting Eq. (15) and $V(0) = J/qn(0)$ into Eq. (10), we obtain

$$C_{crit} = 3k_B T_c . \quad (16)$$

At $C = C_{crit} = 3k_B T_c$, we have

$$V(0) = \left[\frac{3k_B T_c}{m}\right]^{1/2} , \quad (17)$$

$$n(0) = \frac{J}{q\left[\frac{3k_B T_c}{m}\right]^{1/2}} , \quad (18)$$

$$V = \left\{\frac{-q\phi + 3k_B T_c}{m} + \left[\left(\frac{-q\phi}{m}\right)\left(\frac{-q\phi + 6k_B T_c}{m}\right)\right]^{1/2}\right\}^{1/2} , \quad (19)$$

$$\frac{\partial^2\phi}{\partial z^2} = \frac{-4\pi J}{\left\{\frac{-q\phi + 3k_B T_c}{m} + \left[\frac{q\phi}{m}\left(\frac{q\phi - 6k_B T_c}{m}\right)\right]^{1/2}\right\}^{1/2}} . \quad (20)$$

The boundary conditions for Poisson's equation (20) are

$$\phi(0) = 0 = \phi'(0) \quad (21)$$

and

$$\phi(d) = \Phi_d , \quad (22)$$

where prime denotes the derivative with respect to z. Note that $\phi'(0)=0$ is the condition for space-charge-limited emission. For $-q\phi/3k_B T_c << 1$, we may approximate Eq. (20) as

$$\frac{\partial^2\phi}{\partial z^2} \cong \frac{-4\pi J}{\sqrt{3k_B T_c/m}} \quad (23)$$

which gives the dependence of $\phi \propto z^2$ at very small values of z.

It is useful to scale Eq. (20) in terms of the quantities in the cold-fluid C-L flow, i.e.,

$$\frac{\partial^2\hat{\phi}}{\partial\hat{z}^2} = \frac{4\sqrt{2}}{9}\frac{\hat{J}}{\sqrt{\hat{\phi} + 3\hat{T}_c + \left[\hat{\phi}\left(\hat{\phi} + 6\hat{T}_c\right)\right]^{1/2}}} , \quad (24)$$

where

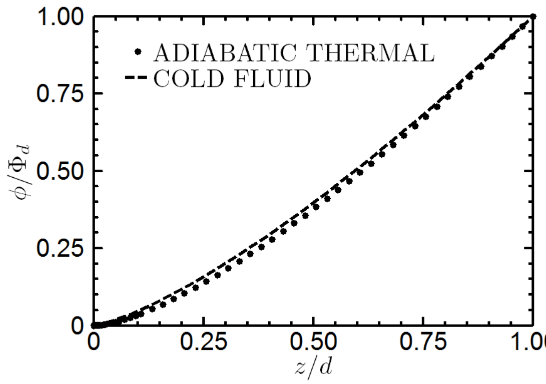

Figure 1: Plots of the normalized potential ϕ/Φ_d versus the normalized distance z/d for a) cold-fluid C-L flow with $\hat{J}=1$ and $\hat{T}_c = 0$ and b) adiabatic thermal C-L flow with $\hat{J}=1.15$ and $\hat{T}_c = 0.001$. For $k_B T_c = 0.1$ eV, the choice of system parameters corresponding to a low voltage diode with $\Phi_d = 100$ V.

$$\hat{\phi} \equiv \frac{\phi}{\Phi_d},\quad \hat{z} = \frac{z}{d},\quad \hat{T}_c = \frac{k_B T_c}{-q\Phi_d},\quad \hat{J} = \frac{J}{J_{CL}} \quad (25)$$

$$J_{CL} = \frac{\sqrt{2}mc^3}{9\pi q d^2}\left(\frac{-q\Phi_d}{mc^2}\right)^{3/2} \quad (26)$$

is the current density of the cold-fluid C-L flow in which $\hat{T}_c = 0$, $\hat{J}=1$ and $\hat{\phi} = \hat{z}^{4/3}$. The boundary conditions for Eq. (23) are:

$$\hat{\phi}(0) = 0 = \hat{\phi}'(0) \text{ and } \hat{\phi}(1) = 1 . \quad (27)$$

Figure 1 shows ϕ/Φ_d versus z/d for a) cold-fluid C-L flow and b) adiabatic thermal C-L flow with $\hat{J}=1.15$ and $\hat{T}_c = 0.001$. For $k_B T_c = 0.1$ eV, the choice of system parameters corresponds to a low-voltage electron diode with $\Phi_d = 100$ V. It is interesting to observe that for the choice of system parameters in Fig. 1, the current density of 1D adiabatic thermal C-L flow is 15% higher than the cold-fluid C-L current given in Eq. (33).

Figure 2 shows ϕ/Φ_d versus z/d for a) cold-fluid C-L flow and b) adiabatic thermal C-L flow with $\hat{J}=2.6$ and $\hat{T}_c = 0.1$. For $k_B T_c = 0.1$ eV, the choice of system parameters corresponds to a low-voltage electron diode with $\Phi_d = 1.0$ V.

SELF-CONSISTENT SIMULATION

One-dimensional self-consistent simulations are performed to verify the theoretical predictions. In the simulations, planar charged sheets are used. A detailed description of the simulation model is given in [9].

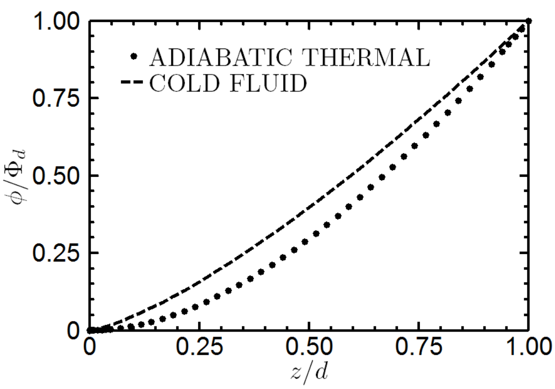

Figure 2: Plots of the normalized potential ϕ/Φ_d versus the normalized distance z/d for a) cold-fluid C-L flow with $\hat{J}=1$ and $\hat{T}_c=0$ and b) adiabatic thermal C-L flow with $\hat{J}=2.6$ and $\hat{T}_c=0.1$. For $k_BT_c=0.1$ eV, the choice of system parameters corresponds to a low-voltage electron diode with $\Phi_d=1.0$ V.

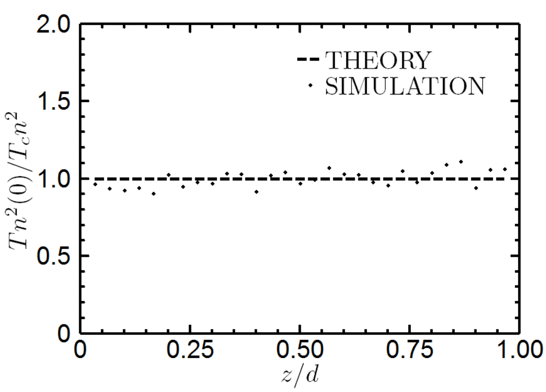

Figure 3: Comparison of $Tn^2(0)/T_cn^2$ versus the normalized distance z/d from simulation with theory for the choice of system parameters corresponding to $k_BT_c/m\omega_p^2d^2=10$, $V(0)/\omega_pd=5.48$ and $-q\Phi_d/m\omega_p^2d^2=10$.

In contrast to the simulations in [9], which employed a truncated Maxwellian distribution in the velocity space at the emitter ($z=0$), the present simulations employ a full Maxwellian distribution with temperature T_c shifted by $V(0)$ in the velocity space at $z=0$. When a charged sheet strikes the plate at $z=d$, it is replaced by a new charged sheet at the emitter ($z=0$). The newly injected charged sheets are taken from the shifted full Maxwellian distribution.

Preliminary simulation results are obtained. To date, the approximation of the adiabatic equation of state, i.e., Eq. (4) or (8), has been found in good agreement with self-consistent simulation in the high-temperature regime with $k_BT_c/m\omega_p^2d^2>>1$. Here, $\omega_p^2=4\pi q^2\bar{n}/m$ is the plasma frequency associated with the average particle density $\bar{n}=d^{-1}\int_0^d n(z)dz$. The plots of $Tn^2(0)/T_cn^2$ versus the normalized distance z/d in Fig. 3 show the comparison between theory and simulation for the choice of system parameters corresponding to: $k_BT_c/m\omega_p^2d^2=10$, $-q\Phi_d/m\omega_p^2d^2=10$, $V(0)/\omega_pd=5.48$. In the simulation, 50000 charged sheets are used. Within the statistical fluctuations, the quantity $Tn^2(0)/T_cn^2$ is conserved. The simulation results in Fig. 3 validate the approximation of the adiabatic equation of state, i.e., Eq. (4) or (8). They also confirm that the normalized (longitudinal) rms thermal emittance of any small segment of the flow is conserved.

CONCLUSION

A theory was presented that describes steady-state one-dimensional Child-Langmuir flow at a self-consistent finite temperature distribution. In particular, warm-fluid equations and adiabatic equation of state were used to derive the self-consistent Poisson equation. The profiles of the charged-particle density, the velocity, the electrostatic potential, the pressure and the temperature are computed. An effort was initiated to validate the theory via self-consistent simulation. To date, good agreement was found between the approximation of the adiabatic equation of state and self-consistent simulation in the high-temperature regime.

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