MAGNETIC FIELD DESCRIPTION IN CURVED ACCELERATOR MAGNETS USING LOCAL TOROIDAL MULTIPOLES

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Abstract

Any introduction on beam dynamics describes the field homogeneity of the accelerator magnets using local derivatives. These are then typically described as plane circular multipoles or 2D harmonics; solutions to the potential equation. The high current operation, foreseen for SIS100 accelerator of FAIR, requires an in detail understanding of the different beam effects, driven by the resonance of the magnets. Therefore different multipole sets were developed and are now finalised in the Local Elliptic Toroidal Multipoles. These are a first order approximation while the plane circular ones are a zero order one in the inverse aspect ratio.

INTRODUCTION

The path of a charged particle beam is deviated in a magnetic field. For uniform dipole fields this path will be a circular trajectory. Standard beam dynamcis then describe higher order distortions by a Taylor series expansion in the transveral coordinates of the tripod moving with the particle. In practise these field distortions are described using 2D harmonics; an approach of sufficient quality as long as the sagitta within the magnet is so small that the resulting field change is negligible.

For machines with a small bending radius this simplifaction is not necessarly valid. Uniform Local Toroidal Multipoles allow presenting the field deterioriation with a concise representation witout the aforementioned artifact.

MULTIPOLE EXPANSIONS FOR STRAIGHT MAGNETS

Plane circular multipoles

The irrotational source-free plane magnetic field is expanded in Cartesian or polar coordinates as a complex field:

$$\begin{aligned}\mathbf{B}(\mathbf{z}) &= B_y(x,y)+iB_x(x,y) = B_0 \sum_{m=0}^{M} \mathbf{C}_m \left(\frac{x+iy}{R_{Ref}}\right)^m \\ &= B_y(r,\theta)+iB_x(r,\theta) = B_0 \sum_{m=0}^{M} \mathbf{C}_m r^m e^{im\theta}.\end{aligned} \quad (1)$$

Plane elliptic multipoles

In a magnet with a rectangular gap an ellipse as reference curve covers a larger area than a circle. So it is advantageous to use elliptic coordinates $x = e\ \cosh\eta\, \cos\psi$ and $y = e\ \sinh\eta\, \sin\psi$, with a, b and $e = \sqrt{a^2 - b^2}$ the major, minor semi-axes and the eccentricity of the reference ellipse, which is expressed in the above coordinates by $\eta = \tanh^{-1}(b/a)$. The Cartesian and the elliptic coordinates are connected by a conformal map:

$$\mathbf{z} = x + iy = e\ \cosh(\eta + i\psi) = e\ \cosh\mathbf{w}. \quad (2)$$

Solving the potential equation by separation leads to hyperbolic functions in η and trigonometric functions in ψ. The complex field expansion is $\mathbf{B}(\mathbf{w}) = B_y(\eta,\psi)+iB_x(\eta,\psi)$:

$$\mathbf{B}(\mathbf{w}) = B_0 \left(\frac{\mathbf{E}}{2} + \sum_{n=1}^{M} \mathbf{E}_n \frac{\cosh[n(\eta+i\psi)]}{\cosh(n\eta_0)}\right).$$

In view of the transformation (2) expansions for the same field are related. In fact:

$$\begin{aligned}&\cosh[n(\eta+i\psi)] = \cosh(n\eta)\cos(n\psi) + i\sinh(n\eta)\sin(n\psi)\\ &= \textstyle\sum_{m=0}^{n} [\mathrm{Re}(t_{m,n}z^m) + i\ \mathrm{Im}(t_{m,n}z^m)]\end{aligned} \quad (3)$$

with the residue

$$t_{mn} = \mathrm{Res}\left(\sinh\mathbf{w}\ \cosh(n\mathbf{w})/\cosh^{m+1}\mathbf{w}\right), \mathbf{w} = i\pi/2 \quad (4)$$

Also from the values for the $\mathbf{E}_n$ values for the $\mathbf{C}_m$ may be found. The latter are not exactly the same as those found from the Euler formulae belonging to the expansions (1) but often give a better approximation.

FIELD EXPANSIONS FOR CURVED MAGNETS

In these cases a torus segment of circular or elliptic cross section is used as reference volume. It is assumed that the field is independent of the toroidal (= azimuthal) variable ϕ. So the potential equation contains only the two variables occurring in the cross section ϕ =const. It cannot be solved by separation of variables but is amenable to approximate R-separation: The Laplacian $\Delta\Phi$ contains a term which accounts for the curvature R_c. This is linear in $\epsilon \propto 1/R_c$, the inverse aspect ratio of the torus. Changing the dependent variable from Φ enable to $\sqrt{h}\Phi$ (where h is proportional to the metric coefficient of the toroidal variable and is a linear polynomial in ϵ) gives the curvature term a dependence on ϵ^2. This may be neglected. The differential operator so truncated is the Laplacian in plane polar or elliptic coordinates and is separable. However, the complex field expansion is no longer possible in such local rotational coordinates. Further the basic vector fields found from gradients of particular solutions for the potential are no longer orthogonal among each other.

Local circular toroidal coordinates

These coordinates are obtained by rotating off-centre dimensionless polar coordinates ρ, ϑ by an angle φ:

$$\begin{array}{lclclcl} X + iY & = & R_C\, h e^{i\varphi}, & Z & = & R_0\, \sin\vartheta, \\ h & = & 1 + \epsilon\, \rho\, \cos\vartheta & \epsilon & = & R_0/R_C \end{array}$$

R_C = major radius = radius of curvature; R_0 = minor radius = reference radius; ϵ the inverse aspect ratio. The Cartesian coordinates X, Y, Z are centred in the torus centre; Z is normal to the equatorial plane.

The approximate solutions of the potential equation obtained by the approximate R-separation are: $\Phi_m = h^{-1/2}\, \rho^m\, e^{im\vartheta}$, $m = 0, 1, 2, \ldots$. Introducing Cartesian coordinates x', y' in the plane $\varphi = \text{const}$:

$$\mathbf{z}' = x' + iy' = R_0\, \rho\, e^{i\vartheta} \tag{5}$$

we get the approximate circular toroidal multipoles:

$$\Phi_m(x',\, y') = \left(\frac{\mathbf{z}'}{R_0}\right)^m - \frac{\epsilon}{4}\left[\left(\frac{\mathbf{z}'}{R_0}\right)^{m+1} + \left(\frac{\mathbf{z}'}{R_0}\right)^{m-1} \frac{|\mathbf{z}'|^2}{R_0^2}\right]. \tag{6}$$

Corresponding (normal and skew) vector fields are (m = 1, 2, ...):

$$\vec{T}_m(x', y') = -\,\tfrac{R_0}{m}\, \nabla' \Phi_m(x', y'),$$
$$\vec{T}_m^{(n)}(x', y') = \mathrm{Re}(\vec{T}_m(x', y')),\;\; \vec{T}_m^{(s)}(x', y') = \mathrm{Im}(\vec{T}_m(x', y')).$$

Local toroidal elliptic coordinates

The coordinates are quite new. They are obtained by rotating off centre elliptic coordinates $\bar\eta$, $\bar\psi$ by an angle $\bar\varphi$:

$$\begin{array}{lclclcl} X + \mathrm{i}Y & = & R_C \bar{h} e^{\mathrm{i}\bar\varphi} & Z & = & R_C \bar{e} \sinh\bar\eta \sin\bar\psi \\ h & = & 1 + \bar{e}\cosh\bar\eta\cos\bar\psi & & & . \end{array}$$

A segment of a torus with elliptic cross section is given by $0 \le \bar\eta \le \bar\eta_0$, $-\pi \le \bar\psi \le \pi$, $-\bar\varphi_0 \le \bar\varphi \le \bar\varphi_0$, with the elliptic aspect ratio $\bar\eta = \frac{\bar{e}}{R_C}$ and the eccentricity $\bar{e} = \sqrt{a^2 - b^2}$.

Apart from a factor $R_C^2 \bar{e}^2 / \left[\cosh(2\bar\eta) - \cos(2\bar\psi)\right]$ the potential equation is:

$$\left[\frac{\partial^2}{\partial\bar\eta^2} + \frac{\partial^2}{\partial\bar\psi^2} - \frac{\bar\epsilon}{\bar{h}}\left(\sinh\bar\eta\cos\bar\psi\frac{\partial}{\partial\bar\eta} + \cosh\bar\eta\sin\bar\psi\frac{\partial}{\partial\bar\psi}\right)\right]\bar\Phi = 0.$$

$$\frac{1}{\sqrt{\bar{h}}}\left[\frac{\partial^2}{\partial\bar\eta^2} + \frac{\partial^2}{\partial\bar\psi^2} - \frac{\bar\epsilon^2}{8\bar{h}^2}\left(\cosh(2\bar\eta) - \cos(2\bar\psi)\right)\right]\left(\sqrt{\bar{h}}\bar\Phi\right) = 0.$$

Approximate multipoles are solutions accurate to the first order in $\bar\epsilon$

$$\bar\Phi_{cn}(\bar\eta, \bar\psi) = \mathcal{S}(\bar\eta, \bar\psi)\cosh(n\bar\eta)\cos(n\bar\psi) + O(\bar\epsilon^2), \tag{7}$$
$$\bar\Phi_{sn}(\bar\eta, \bar\psi) = \mathcal{S}(\bar\eta, \bar\psi)\sinh(n\bar\eta)\sin(n\bar\psi) + O(\bar\epsilon^2), \tag{8}$$

with

$$\mathcal{S}(\bar\eta, \bar\psi) = \left(1 - \bar\epsilon\frac{1}{2}\cosh(\bar\eta)\cos(\bar\psi)\right) \tag{9}$$

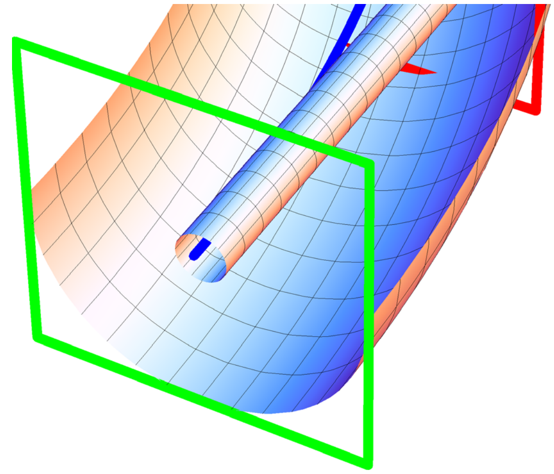

Figure 1: The rotating coil probe within the curved magnet aperture.

The corresponding toroidal components of the magnetic induction are obtained by gradients. These are rather involved in elliptic toroidal coordinates. We prefer to define them in the Cartesian coordinates given in (5)

$$\vec{T}_n^{(n)}(x', y') = -\frac{\bar{e}}{m}\nabla'\bar\Phi_{cm}(x', y') \tag{10}$$
$$\vec{T}_M^{(s)}(x', y') = -\frac{\bar{e}}{m}\nabla'\bar\Phi_{sm}(x', y') \tag{11}$$

$$\vec{B}(x', y') = \sum_{m=1}^{M}\left(\bar{r}_m \vec{T}_n^{(n)}(x', y') + \bar{s}_m \vec{T}_m^{(s)}(x', y')\right)$$

The real, the imaginary part respectively of (7–8) are separately used to replace $\cosh(n\bar\eta)\cos(n\bar\psi)$, $\sinh(n\bar\eta)\sin(n\bar\psi)$, respectively by polynomials in x', y'. Taking into account the factor $S(\bar\eta, \bar\psi)$ leads to the final result:

$$\bar\Phi_{cm}(x', y') = \sum_{m=1}^{n} t_{mn}\mathrm{Re}\left[\Phi_m(x', y')\right] \tag{12}$$

and a corresponding result $\bar\Phi_{sm}(x', y')$ and the imaginary part. The $\Phi_{x',y'}$ are given in (5). The connection between the Fourier components of the rotating coil voltage and the expansions coefficients $\bar{r}_m$, $\bar{s}_m$ can then be done analogously to the case of the circular toroidal multipoles [1].

APPLICATION

The following description expects that the reader has a basic understanding of rotating coil measurements (see e.g. [2]); a longer deviation is given [1]. Here the model of the coil probe in the toroidal field is used to show which effects are to be expected from these new multipoles (see also Fig. 1) . Imagine a pure radial coil with one wire in the rotation centre and the other one at the radius R. Its axis is in the equatorial plane of the torus segment. Then the flux seen by the coil probe at any angle is given by

$$\Phi(\phi) = \int_{r_1}^{r2}\int_{-L}^{L}(\vec{B}\cdot\vec{e}_r)\,\mathrm{d}z\,\mathrm{d}r\,. \tag{13}$$

This flux can then be expressed by

$$\Phi(\phi) = K_n \left[\bar{s}_m C_{mn} \cos(n\phi) - \bar{r}_m D_{mn} \sin(n\phi)\right] \quad (14)$$

using the geometric coefficients of the coil probe K_n (see e.g. [2]). The usual circular multipoles $\mathbf{c_n} = b_n + \mathrm{i} a_n$ can then be given by

$$a_n = -\sum_{m=1}^{M} \bar{s}_m\, C_{mn}, \qquad b_n = \sum_{m=1}^{M} \bar{r}_m\, D_{mn}. \quad (15)$$

Matrices C_{mn} and D_{mn} are equal as long as the coil probes axis is tangential to the bigger torus circle with the middle of the coil the tangential point. The matrices C_{mn} and D_{mn} show how the multipoles correspond to each other. If these matrices had only non zero elements on the diagonal each toroidal multipole would be only be scaled.

Each matrix element of C and D comprises of 2 terms. The conversion matrix (C_{nm}) can be written in the following form

$$C = I - \epsilon \left(U + \mathcal{L}^{co}\right) . \quad (16)$$

The matrix consists of 2 submatrices whose magnitude depend on the ratio $\epsilon = R_0/R_C$. U is given by

$$U = \frac{n}{4(m-1)}\, \delta_{n+1,m} . \quad (17)$$

The matrix $\mathcal{L}^{co}$ is given by

$$\mathcal{L}^{co} = m\, \frac{l^2}{24 R_0^2} - \frac{K_{m+2}}{4\,(m+1)\, K_m} \delta_{n-1,m} . \quad (18)$$

It gives a "feed up" of one multipole to the next. The terms K_m are typically of the same order of magnitude. Thus the first term will be the dominant one for any real coil probe length (typically much larger than the reference radius).

With these results the following conclusions can be drawn: In a gedankenexperiment one could make the term K_m one by using a coil probe with $r_2 = R_0$ and $r_1 = 0$ and the number of windings $= 1/l$. Then even $\lim_{l\to 0} K_m = 1$ would hold. This would be a coil of zero length and thus a coil probe in a plane. For $l = 0$ the first term would be zero and the second term $1/[4(m+1)]$. This shows that the approximate correspondence between plane circular multipoles and circular local toroidal coordinates is a "feed up" and "feed down" of each multipole to his neighbours.

If one uses the coil probe of realistic length (or the length of the dipole magnet) one can estimate the magnitude of the errors made as well as the multipoles affected using the standard 2D plane multipoles instead of the appropriate local toroidal ones.

Magnitude of the terms

The formulae given above were evaluated for the following different machines: the Large Hadron Collider (LHC) at CERN[3], SIS100 [4, 5] and SIS300 at GSI, and NICA [6] at Dubna (see Table 1).

Table 1: Parameters for different machines.

	R_C [m]	R_0 [mm]	ϵ [units]	L [mm]
LHC	2804	17	0.04	600
SIS100	52.5	40	7.62	600
SIS300	52.5	35	6.67	600
NICA	15	40	26.67	600

The parameters given in Table 1 were used to calculate the coefficients of the matrices. Accelerators require a field description with an accuracy of 1 unit and roughly 0.1 unit for the field homogeneity (1 unit equals 100 ppm). Therefore any contribution less than 1 ppm can be ignored.

Due to the circumference of the LHC ϵ is very small and thus the correction of all matrices are very small (less than 1 ppm) except for the matrix $\mathcal{L}^{co}$. It's values close to the diagonal get to a size of 20 units describing a "feed up". Thus only the main field creates a spurious quadrupole of $\approx$ 3 units. All other harmonics are small and thus the spurious ones well below 1 ppm.

For machines with an aspect ratio as found for SIS100 or SIS300 the matrix U is in the order of 100 ppm. It can be neglected except for the main multipole. The values of the matrix $\mathcal{L}^{co}$ get very large. So the standard model of plane circular multipoles is not appropriate for these machines.

CONCLUSION

Local circular and elliptic toroidal multipoles were presented. The modell of a rotating coil probe shows how these multipoles match to the plane circular multipoles commonly used to describe the field distortion in the magnet.

The important ratio is ϵ or $\bar{\epsilon}$ and coorelates the beam size to the radius of curvature. The Gedankenexperiment with a rotating coil probe allows to show that for a infinite thin plane the circular multipoles will smear to the neighbouring toroidal multipoles. A coil with the length of the magnet modells precisely the artifacts to expect if plane circular multipoles are used to descibe the field distortions for a beam with a large sagitta within the dipole.

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