CALCULATION OF HIGH FREQUENCY FIELDS IN RESONANT CAVITIES BASED ON PERTURBATION THEORY*

K. Brackebusch†, H.-W. Glock, U. van Rienen, Institute of General Electrical Engineering, University of Rostock, Germany

Abstract

The knowledge of the eigenmodes of resonant accelerator cavities is essential for the determination of their performance characteristics, comprising resonant frequencies and field distributions inside the cavities. Apart from the material properties the eigenmodes of a cavity depend on its geometry. In spite of the high elaborateness during the complex fabrication process, minor deviations of the actual cavity shape from the desired one are inevitable. Moreover, especially superconducting cavities are subject to extreme operating conditions that may cause deformations of their shape. Any geometry perturbation results in a shift of the resonant frequencies and modified field distributions. In this paper, we will analyse a generalisation of Slater's theorem [1] proposed in literature [2], [3]. The method allows for the calculation of resonant frequencies and field distributions of a slightly perturbed cavity by using a set of precomputed eigenmodes of the unperturbed cavity. We will evaluate the practicability of the method by applying it to cavity geometries for which the eigenmodes are analytically known, ascertain the effort to achieve reasonable calculation results and describe its limitations.

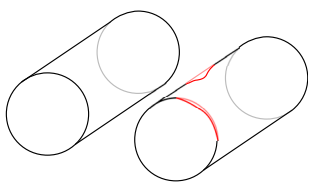

Figure 1: Ideal (left) and actual perturbed cavity (right).

INTRODUCTION

Assuming the geometry and the eigenmodes of a cavity are given, Slater's theorem [1] offers a possibility to calculate the shift of the resonant frequencies which arises if the cavity's geometry is slightly deformed (Fig. 1). The resonant frequency $\tilde{\omega}_\mathrm{i}$ of each mode of the perturbed geometry may be computed by evaluating the following volume integrals

$$\frac{\tilde{\omega}_\mathrm{i}-\omega_\mathrm{i}}{\omega_\mathrm{i}} = \frac{\iiint_{\Delta V}\left(\mu|\vec{H}_\mathrm{i}(\vec{r})|^2-\epsilon|\vec{E}_\mathrm{i}(\vec{r})|^2\right)\,\mathrm{d}V}{\iiint_{V}\left(\mu|\vec{H}_\mathrm{i}(\vec{r})|^2+\epsilon|\vec{E}_\mathrm{i}(\vec{r})|^2\right)\,\mathrm{d}V}. \quad (1)$$

Here $\vec{E}_\mathrm{i}(\vec{r})$, $\vec{H}_\mathrm{i}(\vec{r})$ and ω_i are the stationary electric, the stationary magnetic field and the resonant frequency of the i^{th} mode inside the volume V of the unperturbed cavity. The permittivity and the permeability inside the cavity are denoted with ϵ and μ. ΔV is the volume which is removed by the deformation. Since ΔV has to be a part of V, only volume reductions can be treated. The term in the denominator is identical to four times the mode's energy U_i. Hence, if the energy of all modes is normalized, only the integral over ΔV needs to be evaluated.

However, Slater's theorem (ST) does not allow to determine the field distributions inside the perturbed geometry. In [2] an approach is presented that generalises ST to provide also information about the fields inside the deformed cavity. An almost identical method is introduced in [3]. Knowing also the fields, any further cavity characteristic may be determined. Since no application examples (except a single one in [2]) are known from literature the applicability of the approach shall be examined.

THEORY

The fundamental idea of the procedure denoted in [3] as General Perturbation Theory (GPT) is to derive a certain mode of the perturbed geometry not only from the corresponding unperturbed mode (like in ST) but also from other modes of the unperturbed cavity. The entirety of the stationary fields of modes inside the unperturbed cavity form a system of mutually orthogonal functions [3]

$$\frac{\mu}{2U_\mathrm{i}}\iiint_{V}\vec{H}_\mathrm{i}(\vec{r})\cdot\vec{H}_\mathrm{k}(\vec{r})\,\mathrm{d}V = \delta_\mathrm{ik}, \quad (2)$$

$$\frac{\epsilon}{2U_\mathrm{i}}\iiint_{V}\vec{E}_\mathrm{i}(\vec{r})\cdot\vec{E}_\mathrm{k}(\vec{r})\,\mathrm{d}V = \delta_\mathrm{ik} \quad (3)$$

where δ_ik is the Kronecker Delta. The relationship is valid for all unperturbed modes as long as the integration takes place over the complete volume V. The orthogonality of the modes and the completeness of the set allows to expand each stationary electric and magnetic field $\vec{\tilde{E}}_\mathrm{i}(\vec{r})$, $\vec{\tilde{H}}_\mathrm{i}(\vec{r})$ of the perturbed geometry in terms of the unperturbed fields

$$\vec{\tilde{E}}_\mathrm{i}(\vec{r}) = \sum_{k=1}^{\infty}\alpha_\mathrm{ik}\cdot\vec{E}_\mathrm{k}(\vec{r}), \quad (4)$$

$$\vec{\tilde{H}}_\mathrm{i}(\vec{r}) = \sum_{k=1}^{\infty}\beta_\mathrm{ik}\cdot\vec{H}_\mathrm{k}(\vec{r}). \quad (5)$$

The weighting factors α_ik and β_ik are derived following [3]. Summarising, the interactions of unperturbed modes inside

* Work supported by Federal Ministry for Research and Education BMBF under contracts 05H09HR5 and 05K10HRC

† korinna.brackebusch@uni-rostock.de

the deformed volume part ΔV are analysed and used to approximate the perturbed modes. The interaction terms are similar to the volume integrals used for ST (Eq. 1) but they consider not only the interaction of a mode i with itself but also with every other mode k

$$s_{ik} = 2 \iiint_{\Delta V} \left(\omega_i \mu \vec{H}_i(\vec{r}) \cdot \vec{H}_k(\vec{r}) - \omega_k \epsilon \vec{E}_i(\vec{r}) \cdot \vec{E}_k(\vec{r}) \right) \mathrm{d}V. \tag{6}$$

In [3] the constant factor 2 for the s_{ik} was not mentioned, but it is required for reasons given in [2]. Using the interaction terms s_{ik} two matrices $\underline{\underline{\mathbf{A}}}$ and $\underline{\underline{\mathbf{B}}}$ may be formed with the elements

$$a_{ik} = \omega_i{}^2 \delta_{ik} + \frac{\omega_k}{2U_i} s_{ik}, \tag{7}$$

$$b_{ik} = \omega_i{}^2 \delta_{ik} + \frac{\omega_i}{2U_i} s_{ik}. \tag{8}$$

Calculating the eigenvectors of $\underline{\underline{\mathbf{A}}}$ ($\underline{\underline{\mathbf{B}}}$) finally yields all α_{ik} (β_{ik}). Each eigenvector contains the weighting factors to expand the perturbed stationary field $\vec{\tilde{E}}_i(\vec{r})$ ($\vec{\tilde{H}}_i(\vec{r})$) of exactly one mode. The corresponding eigenvalue is the squared perturbed resonant frequency $\tilde{\omega}_i$ of that mode (which is identical for $\underline{\underline{\mathbf{A}}}$ and $\underline{\underline{\mathbf{B}}}$). Intrinsically only perturbations leading to a reduction of the volume V can be handled. To treat an enlarged perturbed volume $\tilde{V}$, V and likewise its mode set have to be scaled until the scaled volume contains the complete perturbed volume $\tilde{V}$. Afterwards the GPT can be applied as usual.

APPLICATION

For obvious technical reasons the number of unperturbed modes M used to compute the perturbed ones has to be finite. In consequence the expanded modes are only an approximation. To analyse the applicability and accuracy of the method, it was applied to cavity structures where both the unperturbed and the perturbed eigenmodes are analytically known. Therefore uniform radial and longitudinal variations were applied to cylindrical and coaxial cavities. To validate the implementation of the method with common numerical solvers, the set of unperturbed modes was also computed by CST Microwave Studio's Eigenmode Solver AKS [4]. In the following section selected results of a cylindrical cavity are shown.

RESULTS

Analytical Results

Even for a small mode set the perturbed resonant frequencies $\tilde{\omega}_i$ calculated with GPT are significantly more accurate than the results of ST. Fig. 2 shows the relative error of the perturbed resonant frequency (in relation to the actual resonant frequency $\hat{\omega}_i$) depending on the mode order i. It is clearly evident that the error of GPT is more than two orders of magnitude smaller than the error of ST. Likewise, the approximated stationary fields (like it is shown e.g. for

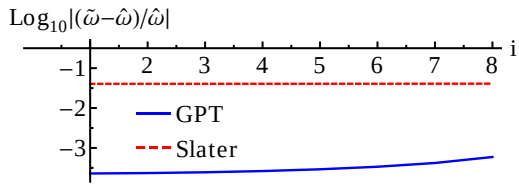

Figure 2: $\Delta R/R = 5\%$, M = 54: The relative error of $\tilde{\omega}_i$ of GPT is less than 10^{-3}, the one of ST is up to 0.05.

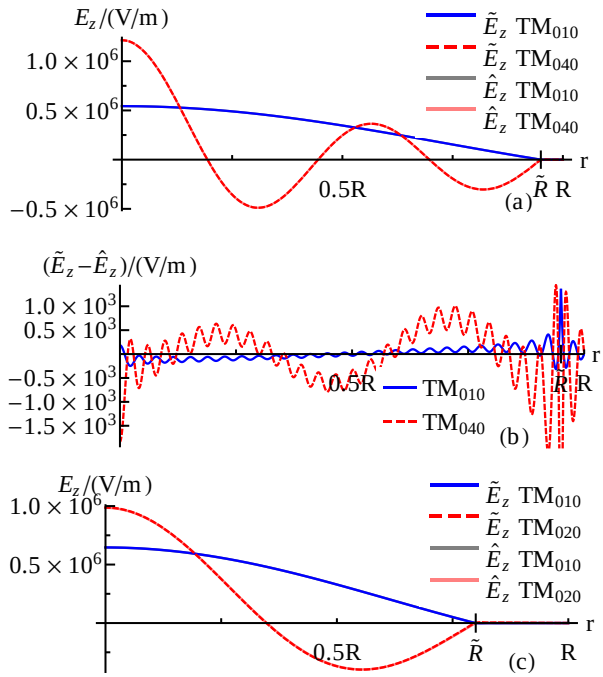

Figure 3: M = 54, $\Delta R/R$=5% (a,b) , $\Delta R/R$=20% (c): (a,c) The field $\tilde{E}_z$ of GPT and the actual field $\hat{E}_z$ coincide so well that only two lines can be observed. (b) The deviation of $\tilde{E}_z$ along the radius r is almost everywhere less than 10^{-3} of its maximal value. Close to the perturbed boundary it increases.

the longitudinal electrical field $\vec{\tilde{E}}_z(\vec{r})$ in Fig. 3) match the actual ones ($\vec{\hat{E}}_z(\vec{r})$) excellently. Even for substantial perturbations the fields coincide very well (Fig. 3(c)). For modes of lower order the deviation between $\tilde{\omega}_i$ and $\hat{\omega}_i$ converges towards zero as the size M of the mode set increases (Fig. 4). A set of 50 unperturbed modes already provides results with a relative deviation of less than 10^{-3}.

Considering modes of higher order a serious problem occurs. The error of $\tilde{\omega}_i$ increases step-like if a certain mode order i_{crit} is reached (Fig. 5). Any results incipient from the critical mode are unsuitable since the relative error lies in the range of the relative geometry variation. This abrupt loss of accuracy is caused by the interaction terms s_{ik}. For both ST and GPT each s_{ik} constitutes an approximation where unknown parameters of the perturbed cavity are substituted by the unperturbed ones. If the order (or rather the frequency) of one mode reaches i_{crit}, the signs of the numer-

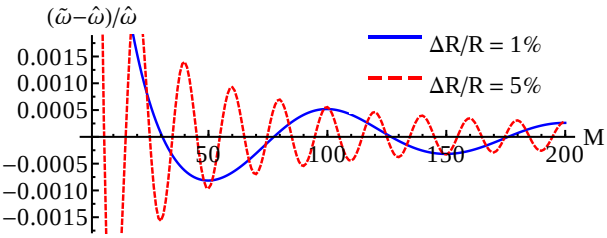

Figure 4: The relative error of $\tilde{\omega}_{\mathrm{TM}\,010}$ converges with a similar decay towards zero for different radius reductions (depending on the size M of the mode set).

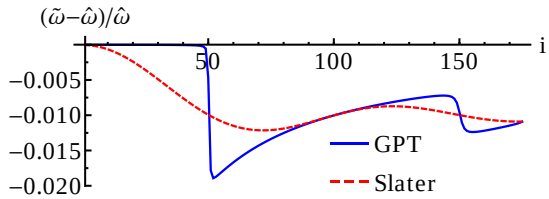

Figure 5: $\Delta R/R = 1\%$: At the 50^{th} mode the error of GPT increases abruptly, so that the results become invalid.

ators of Eq. 1 and 6 change irregularly. (In specific cases the numerators may also become zero.) This leads to incorrect results. The critical value $\mathrm{i}_{\mathrm{crit}}$ and thereby the maximal number N_{max} of modes with reasonable results are only determined by the perturbed volume part ΔV. Increasing the size M of the mode set does not improve this limitation at all. For one-dimensional perturbations of coaxial and cylindrical cavities it can be proven that

$$\mathrm{i}_{\mathrm{crit}} = \left\lceil \frac{\xi}{2\Delta\xi} \right\rceil, \tag{9}$$

$$N_{\mathrm{max}} = \left\lceil \frac{\xi}{2\Delta\xi} \right\rceil - 1. \tag{10}$$

Here $\frac{\Delta\xi}{\xi}$ is the relative one-dimensional reduction of the cavity. Hence, for geometry variations of e.g. 10% only 4 modes can be determined.

Numerical Results

The use of numerically computed unperturbed modes is related with some difficulties. Since only the stationary fields in the (small) volume part ΔV are used for the interaction terms s_{ik}, the grid resolution needs to have an appropriate density to reproduce the fields adequately accurate. As it is only possible to compute the entire fields a high overhead concerning the computational effort and the data amount is unavoidable. A more serious problem is the fact that ΔV is a volume at the cavity's boundary. Hence the fields inside ΔV may contain serious inaccuracies due to the geometry error. This is illustrated on the example of the finite integration technique in Fig. 6. If the whole volume V were considered the geometry error would scarcely effect the results, but if considering only ΔV its influence

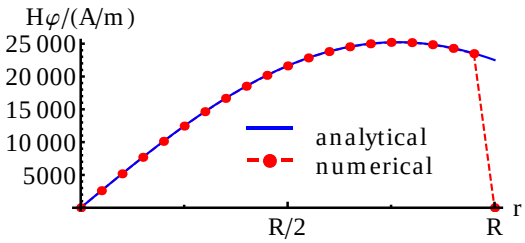

Figure 6: Unperturbed TM_{010} mode of cylindrical cavity: The numerically computed field is forced to zero at the boundary since grid points and boundary do not coincide. Thus, it differs considerably from the analytical field.

is very high. By extrapolating the field close to the boundary the error can be reduced. In doing so the field components normal and tangential to the boundary have to be determined and treated separately because of the different boundary conditions. However, this requires additional effort. Therefore it is intended to use numerical techniques that are more suited to compute the boundary fields.

CONCLUSIONS

The General Perturbation Theory as described in [2], [3] is suitable to calculate very accurate results for the resonant frequency and the stationary field of the eigenmodes of a perturbed cavity. The accuracy increases by the size of the set of unperturbed modes used for the computation. Already for a small set of modes, the method supplies much better results for the resonant frequencies than Slater's theorem does. However, the computation of usable results is limited to a maximal mode order, which decreases with increasing geometrical perturbation. Therefore, the method's application range is restricted in the extent of the perturbation and/or the number of ascertainable modes. The use of common numerical solvers to determine the unperturbed set of modes is not straightforward, since the numerical results have to reproduce the fields adequately close to the boundary, while many numerical methods hold inaccuracies in particular in that region.

REFERENCES

[1] J.C. Slater, *Microwave Electronics*, Reviews of Modern Physics, Vol. 18, No. 4, October 1946.

[2] G. E. Dombrowski, *Matrix formulation of Slater's cavity perturbation theorem*, J. Appl. Phys. 55, 2648 (1984).

[3] D. Meidlinger, *A General Perturbation Theory for Cavity Mode Field Patterns*, Proceedings of SRF2009, Berlin, Germany, THPPO005, 2009.

[4] Microwave Studio ®, CST AG, 64289 Darmstadt, Germany.