Theory of Microwave Instability
and Coherent Synchrotron Radiation in Electron Storage Rings

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Outline

- Motivation
- Brief History
- Coasting Beam Theory
- CSR Wake and Impedance
- Bunch Beam Theory
- Comparison with Experiments
- Conclusion

Luminosity in e^+e^- Colliders

As one of the most important parameters in a collider, its bunch luminosity can be written as,

$$L_b = \frac{f_{rev} N_b \gamma}{2 r_e} \left(\frac{\xi_y}{\beta_y^*}\right) R_{hourglass},$$

where f_{rev} is the revolution frequency, N_b bunch population, γ the Lorentz factor, r_e the electron classic radius, β_y^* beta function at the collision point, ξ_y beam-beam parameter, and $R_{hourglass}$ geometrical reduction factor.

To increase the luminosity, one wants:

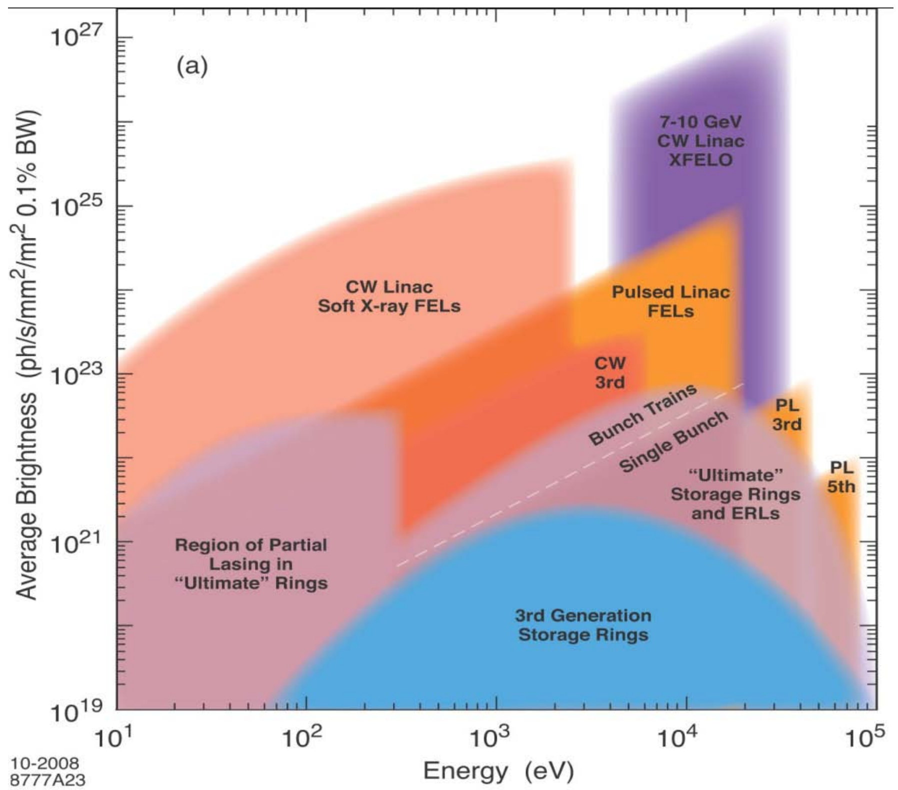
- $I_b = e f_{rev} N_b$ as high as possible but limited by instabilities
- β_y^* as small as possible but limited by the hourglass effect. As a result, β_y^* approximately equals σ_z in colliders

Spectral Brightness

Due to diffraction of light, the spectral brightness from spontaneous synchrotron radiation in an undulator is limited to,

$$B_\lambda = \frac{F_\lambda}{(\lambda/2)^2},$$

where F_λ is the photon flux and λ is the wavelength of x-ray. To go beyond this limit, one needs the process of FEL. That implies much higher peak current or much shorter bunch length.



(courtesy of Bob Hettel)

Coherent THz Radiation

F. Sannibale *et al.* PRL (93) 094801 (2004)

Total radiation of an electron bunch with a population of N,

$$\frac{dP}{d\omega}_{TOT} = \frac{dP}{d\omega}_{INC}\,[N + N(N-1)\,|\,f(\omega)\,|^2],$$

where f(ω) is the Fourier transform of the normalized longitudinal distribution.

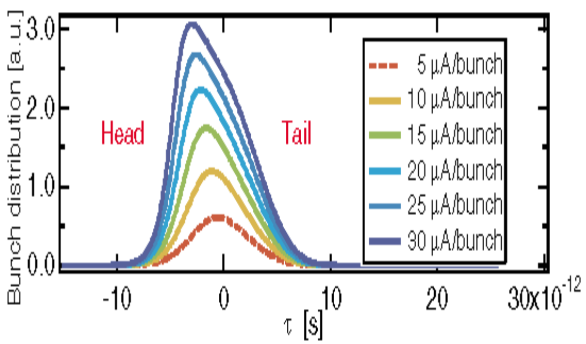


FIG. 1 (color online). Calculated equilibrium longitudinal distribution for different currents per bunch using the shielded SR wake. BESSY II case with a natural bunch length of 2.5 ps.

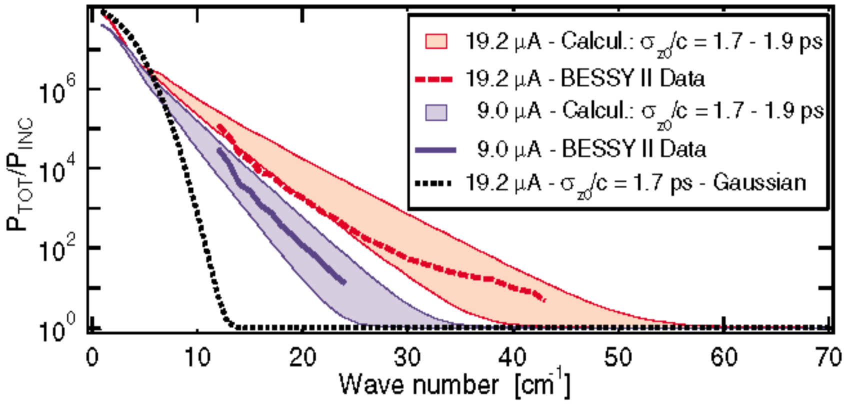


FIG. 2 (color online). CSR gain as a function of the wave number $1/\lambda$. The BESSY II data for two different currents per bunch are compared with the shielded SR calculation and with the curve for a Gaussian distribution of the same length.

Brief History of Longitudinal Instability

- Theory
  - Valsov, equation for collective effects, 1945
  - Landau, Landau damping (poles), 1946
  - Neil, Sessler, dispersion relation (coasting beam), 1965
  - Keil, Schnell, instability criterion (coasting beam), 1969
  - Haissinski, equilibrium, nonlinear integral equation, 1973
  - Boussard, instability criterion (bunch beam), 1975
  - Sacherer, mode-coupling (bunch beam), 1977
  - Suzuki, Chin, and Satoh, Gaussian, 1983
  - Oide, Yokoya, incoherent spectrum, Haissinski,1990
- Simulation
  - Bane, Zotter, …, Haissinski solver and micro-particle, 1985
  - Warnock and Ellison, robust Haissinski and VFP solver, 2000

Dispersion Relation
for coasting beam

From the linearized Vlasov equation of coasting beam, one can derive

$$1 = i\frac{cI}{\alpha\gamma\sigma_\delta^2 I_A}\left(\frac{Z(k)}{k}\right)\int_{-\infty}^{\infty}\frac{dF_0/dp}{p-a}dp$$

Where Z(k) impedance per unit length and is F_0 is a Gaussian distribution

$$F_0 = \frac{1}{\sqrt{2\pi}}e^{-p^2/2}$$

In general, $a=\omega/c\alpha k\sigma_\delta$, is a complex number and is to be solved. As one can see, there is a pole on the real axis. The correct treatment of the pole leads to Landau damping. Actually, one can evaluate the integral in the upper half plane and then analytically continue it into the lower half plane. The correct result of the integral is given

$$G(a) = -1 + \sqrt{\frac{\pi}{2}}ae^{-a^2/2}(erfi[\frac{a}{\sqrt{2}}] - i)$$

Beam is unstable if $\mathrm{Im}[\omega] > 0$.

CSR Wakefield and Impedance in Free Space

Wakefield due to CSR was given by Murphy, Krinsky, and Gluckstern in 1997,

$$W(z) = \frac{4\pi\rho^{1/3}}{3^{1/3}} \frac{\partial}{\partial z} z^{-1/3} = -\frac{4\pi\rho^{1/3}}{3^{4/3}\sigma_z^{4/3}} q^{-4/3}$$

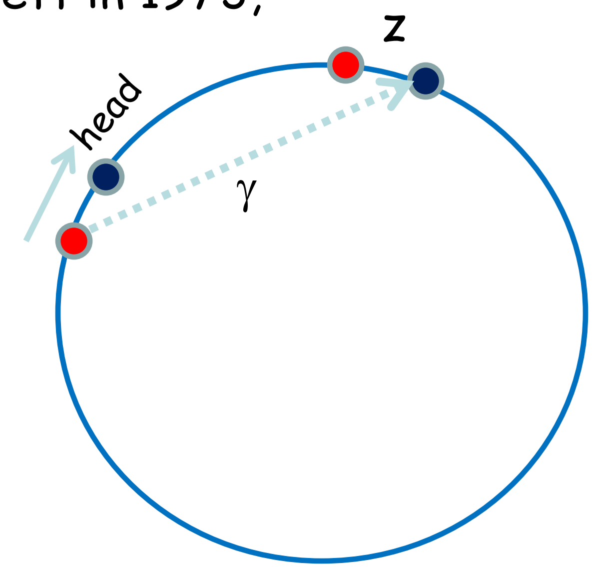
For z>0. It vanishes when z<0 (force is acting on the electron ahead).

Impedance was derived by Faltens and Laslett in 1973,

$$Z_{CSR}(k) = (\frac{4\pi}{c}) \frac{\Gamma(\frac{2}{3})(\frac{\sqrt{3}}{2} + \frac{i}{2})}{3^{1/3}} (\rho k)^{1/3}$$

where ρ is the bending radius.

It does not depend on energy (universal) but a shorter bunch makes it worse.



Microbunching Instability

G. Stupakov and S. Heifets, PRSTAB **5**, 054402 (2002)

By studying a perturbation

$$\psi_1 = \hat{\psi}(\delta) e^{-i\omega s/c + ikz}$$

Found the beam becomes unstable if

$$k\rho < 2.0\Lambda^{3/2}$$

where

$$\Lambda = \frac{I}{\alpha\gamma\sigma_\delta^2 I_A}$$

I: beam current
I_A: Alfven current, 17045 A
α: momentum compaction
γ: Lorentz factor
σ_δ: relative energy spread

Dispersion relation

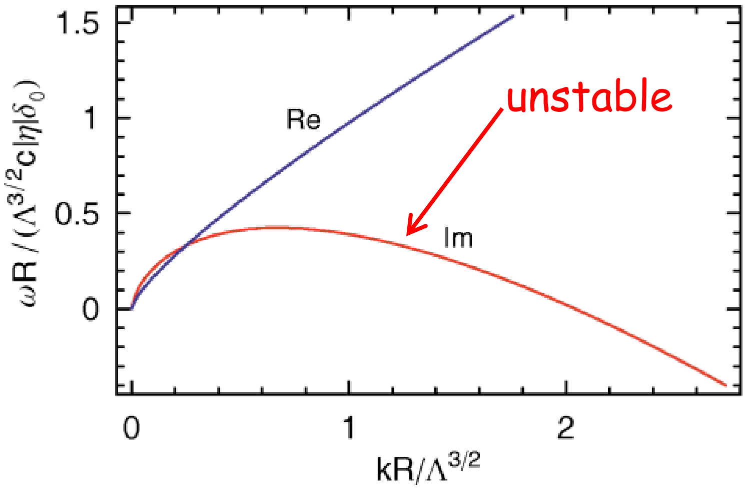


FIG. 1. (Color) The imaginary (Im) and real (Re) parts of the frequency ω as functions of $kR/\Lambda^{3/2}$, for a positive value of η. For negative values of k, the frequency can be found from the relation $\omega(-k) = -\omega^*(k)$ which follows from Eq. (9).

Experimental Observation of CSR Instability at ALS

(J. Byrd *et al*, PRL **89**, 224801, 2002)

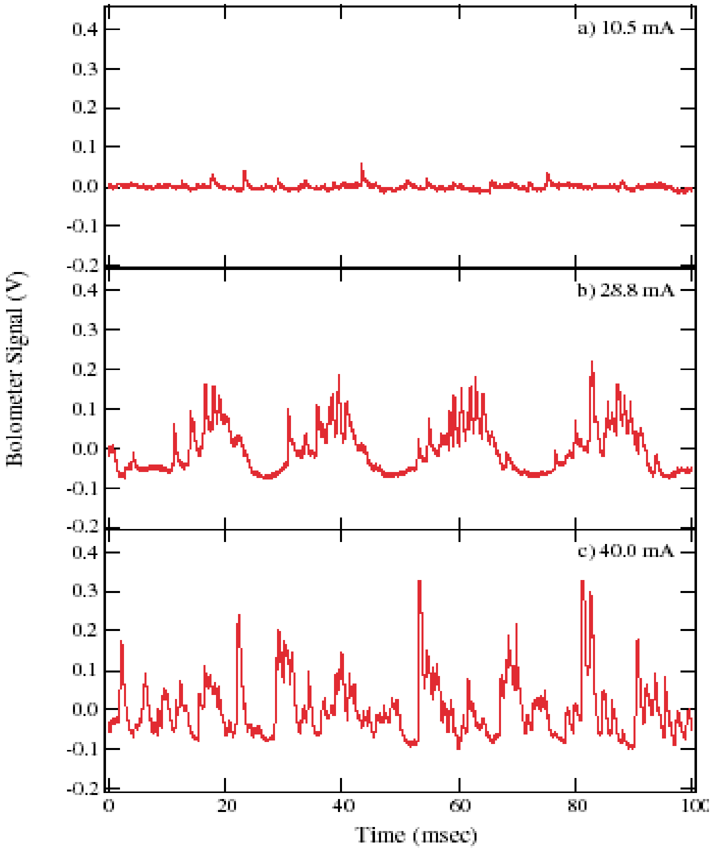


FIG. 1 (color online). Bolometer signal measured demonstrating bursting above threshold at three current values. Between 27 and 31 mA the bursts develop a periodic behavior. Above this current they appear more chaotic.

Coasting beam theory: G. Stupakov and S. Heifets, PRST-AB, **5**, 054402, (2002).

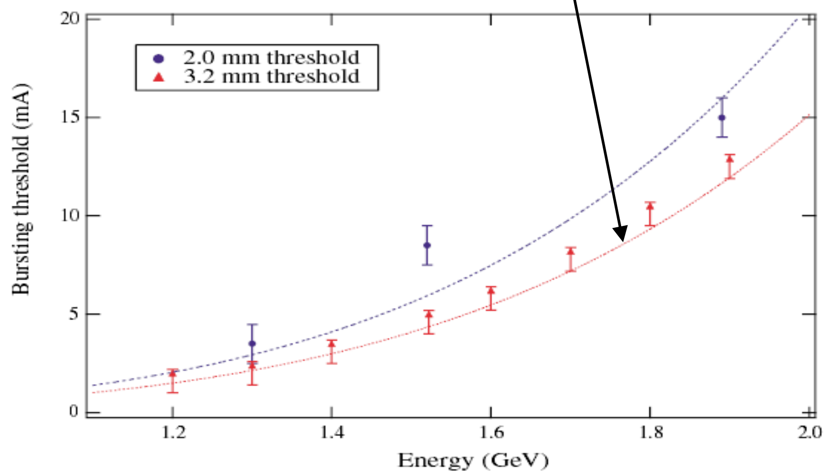


FIG. 4 (color online). Bursting threshold as a function of electron beam energy at 3.2 and 2 mm wavelengths. Data are shown as points. Calculated threshold using nominal ALS parameters at 3.2 and 2 mm wavelengths are shown as dashed lines.

Unstable if

$$I_b > \frac{\pi^{1/6}\alpha\gamma\sigma_\delta^2 I_A \sigma_z}{\sqrt{2}\rho^{1/3}\lambda^{2/3}}$$

CSR with Shielding by Parallel Plates

An impedance with scaling property is given by

$$\left(\frac{\rho}{h}\right)\left(\frac{Z(n)}{n}\right)_P = (2\pi^2)2^{4/3}(\frac{4\pi}{c})[n(\frac{h}{\rho})^{3/2}]^{-4/3}\sum_{p>=1,3,...}[Ai'(u)(Ai'(u)-iBi'(u))+uAi(u)(Ai(u)-iBi(u))]$$

where h is the distance between two plates, n=kρ, Ai and Bi are Airy functions, and their argument u is defined as

$$u = \frac{\pi^2 p^2}{2^{2/3}}[n(\frac{h}{\rho})^{3/2}]^{-4/3}$$

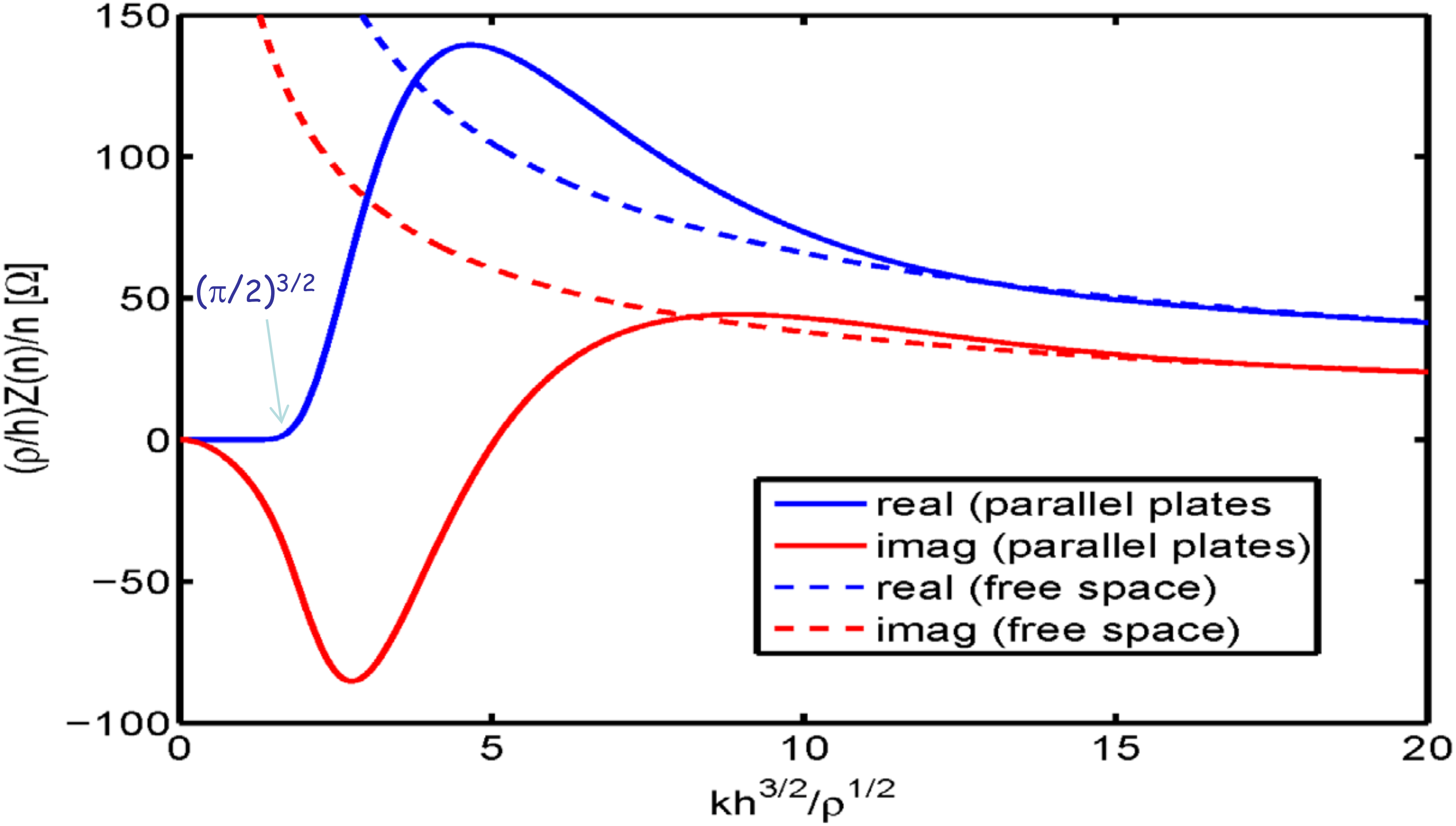
Dependence of n is all through

$$n(\frac{h}{\rho})^{3/2}$$

In fact, this scaling property holds for the CSR impedance in free space, formally

$$\left(\frac{\rho}{h}\right)\left(\frac{Z(n)}{n}\right)_{CSR} = (\frac{4\pi}{c})(\frac{\Gamma(2/3)}{3^{1/3}})(\frac{\sqrt{3}}{2}+\frac{i}{2})[n(\frac{h}{\rho})^{3/2}]^{-2/3}$$

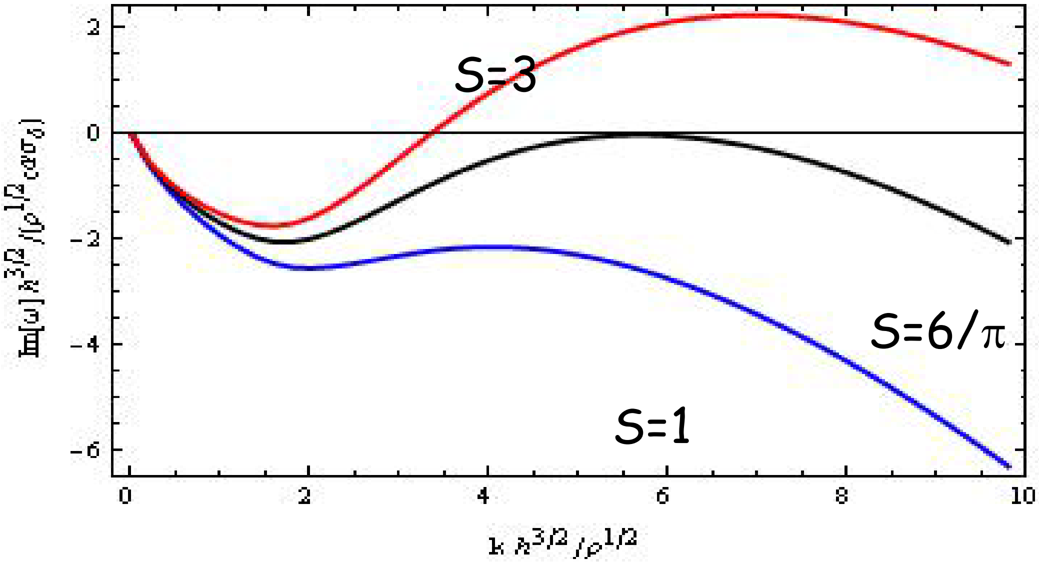
Scaling and Asymptotic Properties



The scale defines the strength of impedance and the location of the peak defines where the shielding effects start.

Coasting Beam Theory
for CSR with Parallel Plates

Dispersion relation



A scaled current:

$$S = \frac{Ih}{\alpha\gamma\sigma_\delta^2 I_A \rho}$$

- o The threshold is $S^{th} = 6/\pi$
- o first unstable mode is $k=5.7\rho^{1/2}/h^{3/2}$.

Applying to a bunched beam, the beam is unstable if

$$I_b > \frac{3\sqrt{2}}{\pi^{3/2}} \alpha\gamma\sigma_\delta^2 I_A \frac{\sigma_z}{h}$$

Note it does not depend on ρ.

CSR in NSLS VUV Ring

Carr, Kramer, Murphy, Lobo, and Tanner

Nucl. Instrum. Methods Phys. Res. Sect. A 463, 387-392 (2001)

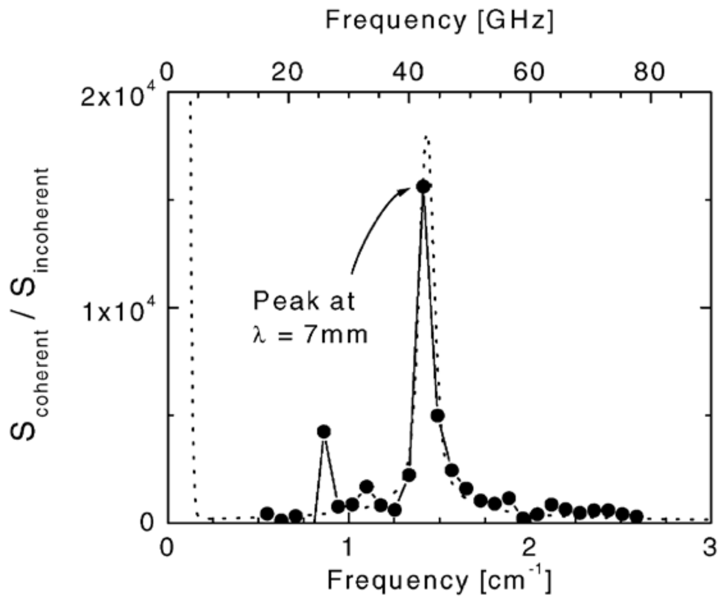


Fig. 4. (Symbols and solid line): measured spectral content of the coherent emission, relative to the incoherent synchrotron spectrum (ratio of spectra shown in Fig. 3). The small peak near 0.8 cm^{-1} is probably due to poor S/N for the incoherent spectrum. Dotted line: modeled spectral content, including a non-observable Gaussian component, centered at zero frequency, for the overall bunch shape.

A Comparison with the Measurement

Parameter	Measurement	Theory
Threshold wavelength	7.0 mm	6.9 mm
Threshold current	100 mA	134 mA

- Bunch length: σ_z = 5 cm.
- Bending radius: ρ =1.9 m
- Full gap: h = 4.2 cm.
- The shielding parameter: $\chi=\sigma_z\rho^{1/2}/h^{3/2}$ = 8.

Synchrotron Oscillation

Electrons execute synchrotron oscillation at frequency $f_s=\nu_s f_{rev}$. The synchrotron tune is given by

$$\nu_s = \sqrt{\frac{\alpha f_{rf}}{2\pi f_{rev}}\left(\frac{eV_{RF}}{E_0}\right)\cos\phi_s}$$

where $E_0 = \gamma mc^2$ is the beam energy. The equilibrium is a Gaussian with a bunch length

$$\sigma_z = \frac{\alpha c \sigma_\delta}{\omega_s}$$

where $\omega_s=2\pi f_s$ and V_{rf} is necessary to compensate the energy loss U_0

$$U_0 = eV_{RF}\sin\phi_s$$

Longitudinal Beam Dynamics

Hamiltonian is given as

$$H = \frac{1}{2}(q^2 + p^2) - I_n \int_{-\infty}^{q} dq'' \int_{-\infty}^{\infty} dq' \lambda(q') W(q'' - q')$$

where I_n is the normalized current introduced by Oide and Yokoya (1990)

$$I_n = \frac{r_e N_b}{2\pi \nu_s \gamma \sigma_\delta} = \frac{\sigma_z I_b}{\alpha \gamma \sigma_\delta^2 I_A}$$

$q = z/\sigma_z$, $p = -\delta/\sigma_\delta$ and $W(q)$ is the integrated wake per turn. The independent variable is $\theta = \omega_s t$.

Vlasov-Fokker-Planck equation is written as

$$\frac{\partial \Psi}{\partial \theta} - \{H, \Psi\} = 2\beta \frac{\partial}{\partial p}(p\Psi + \frac{\partial \Psi}{\partial p})$$

where $\Psi(q,p;\theta)$ is the beam density in the phase space and $\beta = 1/\omega_s \tau_d$.
A robust numerical solver was developed by Warnock and Ellison (2000).

Haissinski Distributions

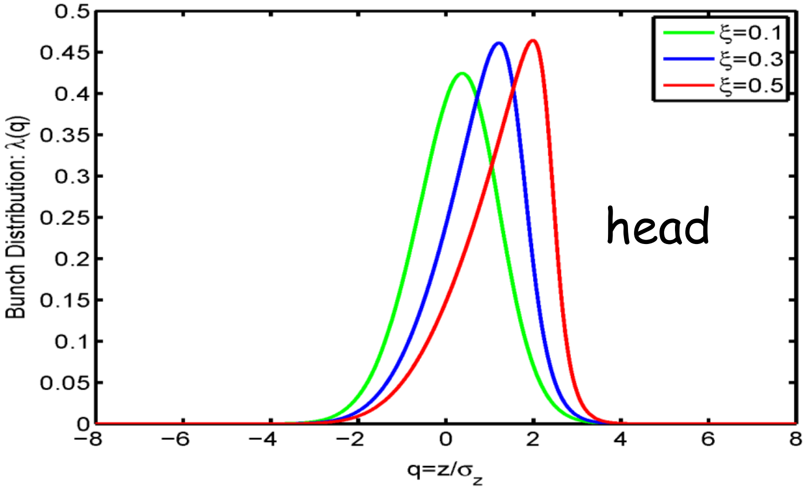
It is easy to see that

$$\psi_0(q,p) = \frac{1}{\kappa\sqrt{2\pi}} \exp[-H] = \lambda(q)\exp[-\frac{p^2}{2}]/\sqrt{2\pi}$$

is a static solution of the VFP equation provided that λ(q) satisfies the Haissinski integral equation

$$\lambda(q) = \exp[-\frac{q^2}{2} + I_n \int_{-\infty}^{\infty} dq'\lambda(q')S(q-q')]/\kappa$$

where S(q) is an integral of W(q) and given as $S(q) = \int_{-\infty}^{q} dq' W(q')$



- A scaled current: $\xi = I_n \rho^{1/3}/\sigma_z^{4/3}$ for CSR in free space
- The threshold is ξ^{th} = 0.5 as we will see later

Generalized Sacherer Integral Equation

To study a perturbation near Ψ_0, we expand Ψ_1 as,

$$\psi_1(\phi, K;\theta) = \sum_{l=-\infty}^{\infty} R_l(K) e^{il\phi} e^{-i\frac{\Omega}{\omega_s}\theta}.$$

Turn the linearized Valsov equation to a set of integral equations

$$\left(\frac{\Omega}{\omega_s} - l\frac{\omega(K)}{\omega_s}\right) P_l(K) = \sum_{m=-\infty}^{\infty} \int_0^{\infty} dK' G_{l,m}(K,K') P_m(K')$$

where the kernel is

$$G_{l,m}(K,K') = \frac{-\sqrt{2} l I_n c e^{-(K+V_{\min})}}{\sqrt{\pi}\kappa\sigma_z} \mathrm{Im}\left[\int_0^{\infty} d\nu \frac{Z(\nu/\sigma_z)}{\nu} h_l(\nu,K) h_m^*(\nu,K')\right]$$

and

$$h_l(\nu,K) = \int_0^{2\pi} \frac{d\phi}{2\pi} e^{-il\phi + i\nu q(\phi,K)}.$$

When Im[Ω] > 0, the beam is unstable.

Laguerre Polynomial Expansion

We decompose

$$P_l(K) = e^{-K} \sum_{\alpha=0}^{\infty} a_l^{\alpha} f_{\alpha}^{(l)}(K)$$

where

$$f_{\alpha}^{(|l|)}(K) = \sqrt{\frac{\alpha!}{(|l|+\alpha)!}} K^{|l|/2} L_{\alpha}^{|l|}(K)$$

Laguerre polynomials

Using the orthogonal-normal condition of the polynomials, we reduce the Sacherer integral equation to a set of linear equations

$$\frac{\Omega}{\omega_s} a_{\alpha}^{l} = \sum_{m=-\infty}^{\infty} \sum_{\beta=0}^{\infty} M_{l,m}^{\alpha,\beta} a_{\beta}^{m}$$

Clearly, it is an eigen value problem. Ω/ω_s is the eigen value. In fact, M is a real matrix. When current is small, all eigen values are real and therefore the beam is stable. It becomes unstable, when the first pair of complex eigen value emerge as the current increases.

Matrix Elements

The matrix elements are given by

$$M_{lm}^{\alpha\beta} = l(\delta_{lm} O_{(l)}^{\alpha\beta} - C_{lm}^{\alpha\beta})$$

and

$$O_{(l)}^{\alpha,\beta} = \int_0^\infty dK \frac{\omega(K)}{\omega_s} e^{-K} f_\alpha^{(l)}(K) f_\beta^{(l)}(K)$$

$$C_{lm}^{\alpha\beta} = \frac{\sqrt{2} I_n c e^{-V_{\min}}}{\sqrt{\pi \kappa \sigma_z}} \mathrm{Im}[\int_0^\infty d\nu \frac{Z(\nu/\sigma_z)}{\nu} g_l^\alpha(\nu) g_m^\beta(\nu)^*]$$

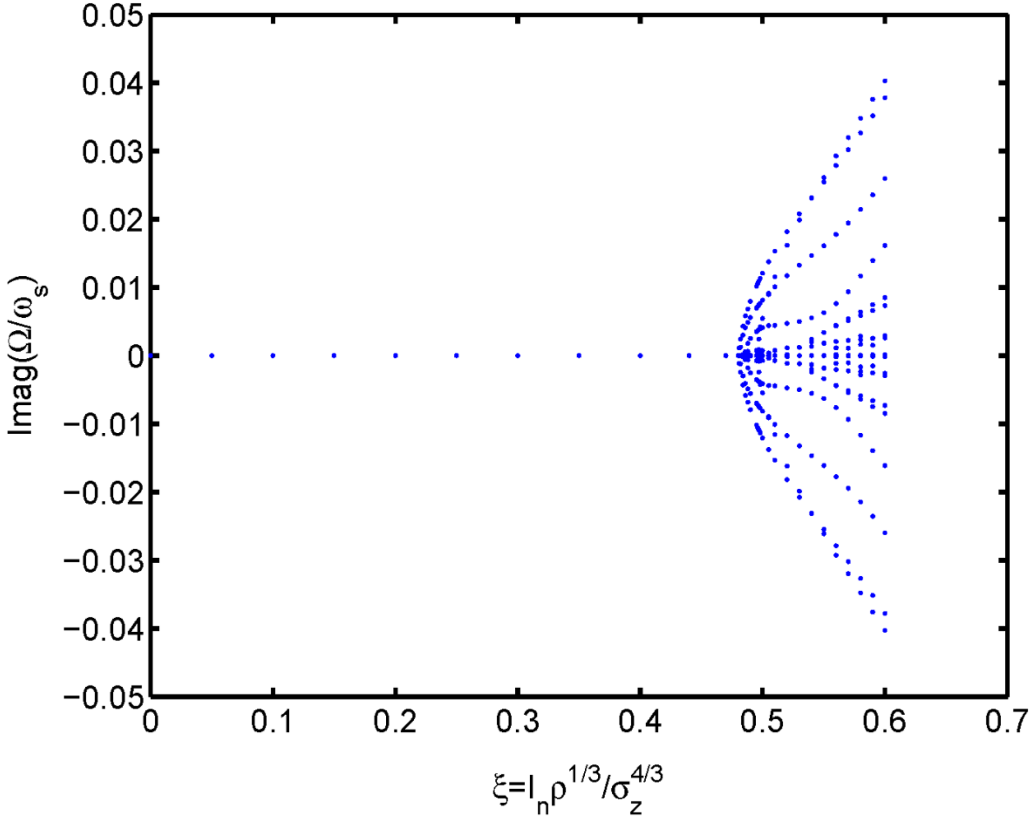
and

$$g_l^\alpha(\nu) = \int_0^\infty dK e^{-K} f_\alpha^{(l)}(K) h_l(\nu, K)$$

One needs to evaluate these integrals for the matrix elements.

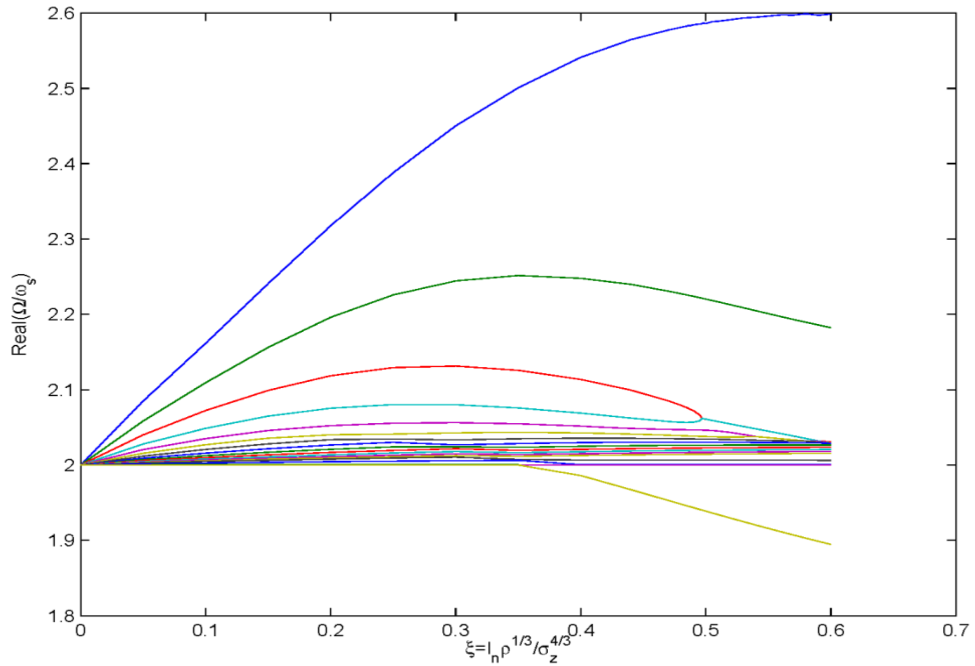
Imaginary Part of Eigen Values

Y. Cai, PRSTAB 14, 061002 (2011)



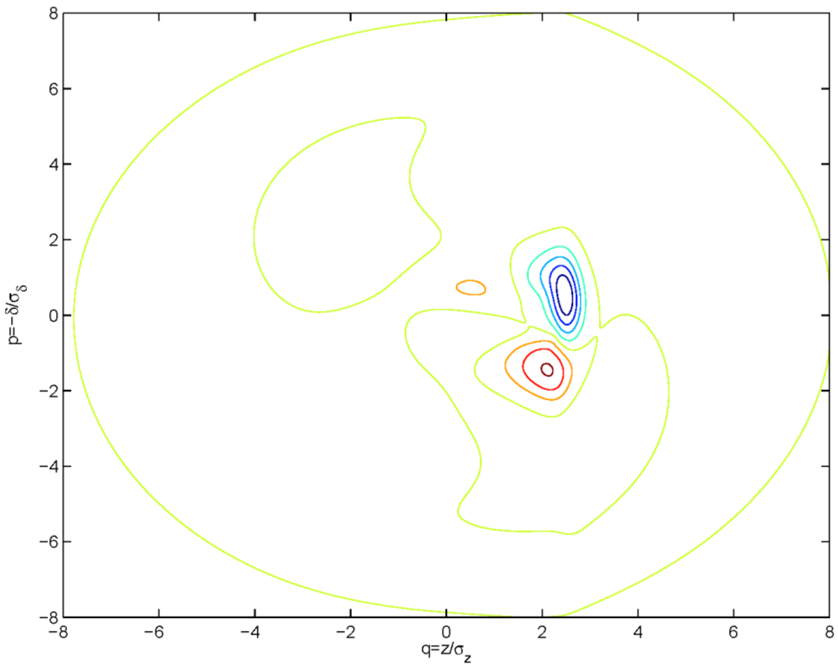
For CSR in free space, we found the threshold $\xi^{th=}0.482$.

Mode Coupling for Quadrupole Modes
for CSR in Free Space



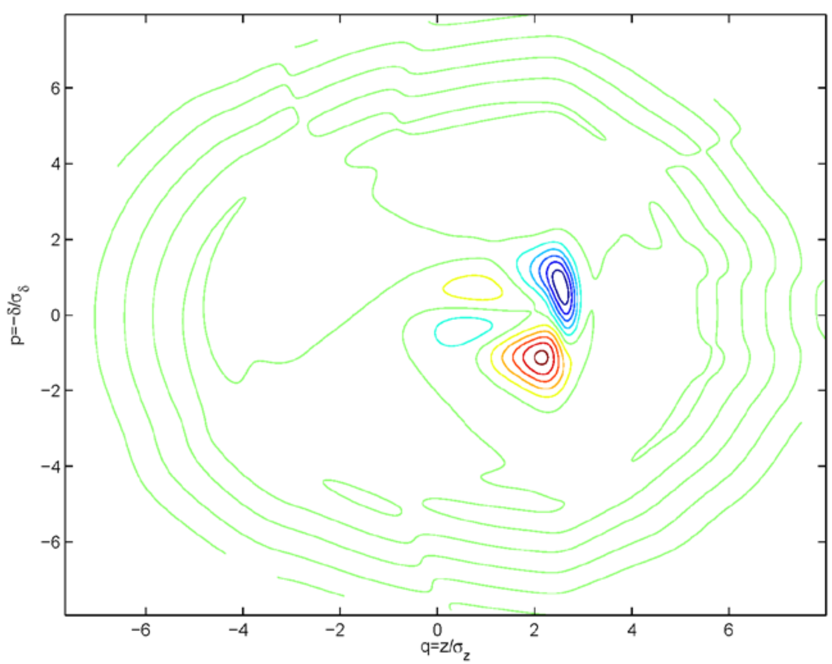
ξ^{th} = 0.5, Y. Cai, PRSTAB **14**, 061002 (2011). The same result was obtained also with simulations, K. Bane, Y.Cai, and G. Stupakov, PRSTAB **13**, 104402 (2010).

Comparison of Unstable Modes
for CSR in Free Space



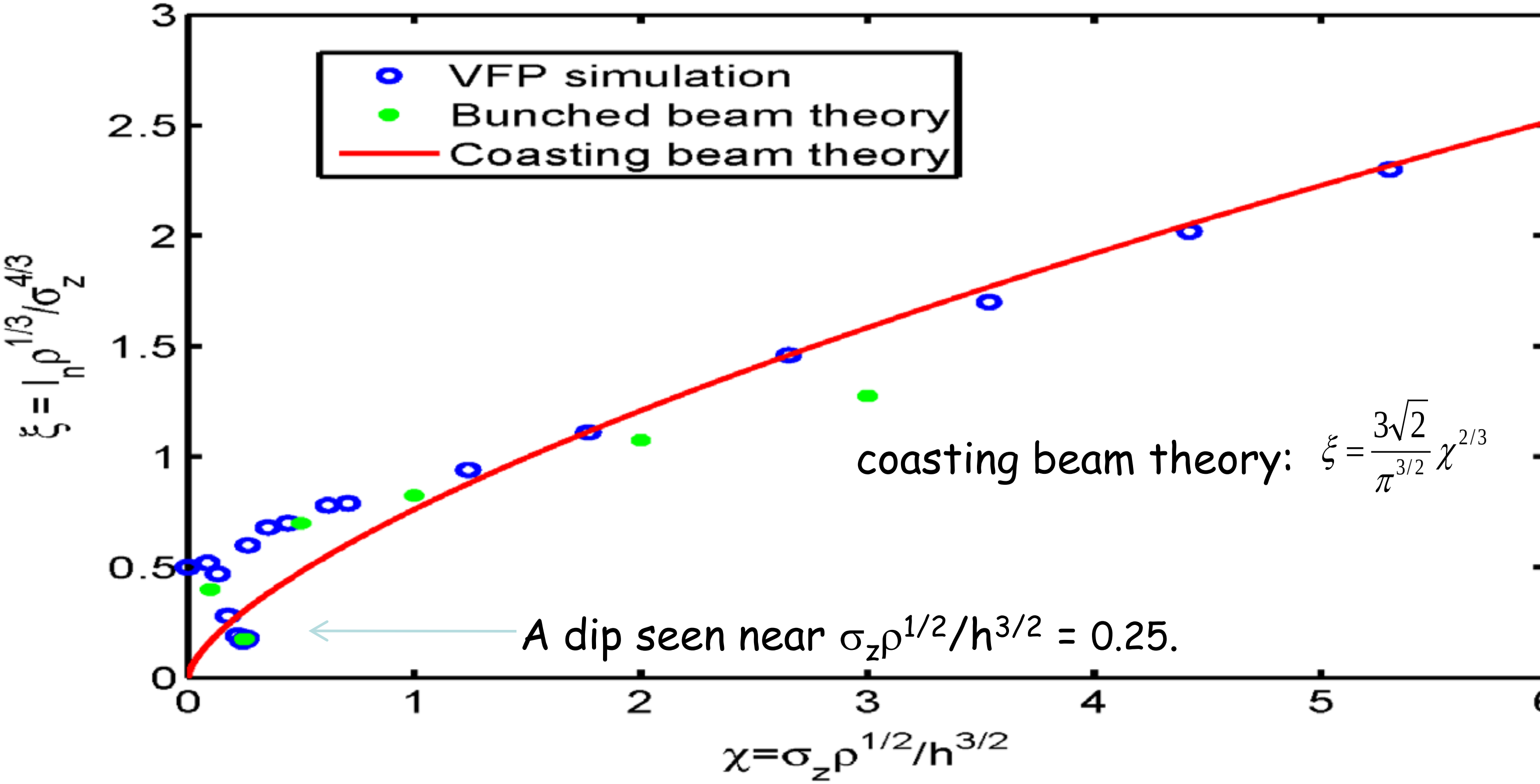
VFP Simulation , 2β=0.0032

At a higher current, ξ=0.560.



Calculation using
Laguerre polynomials
with the highest growth rate

Threshold of Instability
for CSR of Parallel Metal Plates



Threshold ξ^{th} becomes a function of the shielding parameter $\chi = \sigma_z\rho^{1/2}/h^{3/2}$. Simulation was carried out by K.Bane, Y.Cai, and G. Stupakov, PRSTAB **13**, 104402 (2010). For a long bunch, the coasting beam theory agrees well with the VFP simulation.

Bursting Thresholds at BESSY II and MLS

Wustefeld et al. IPAC'10 p. 2508 (2010)

MLS:

Measured thresholds

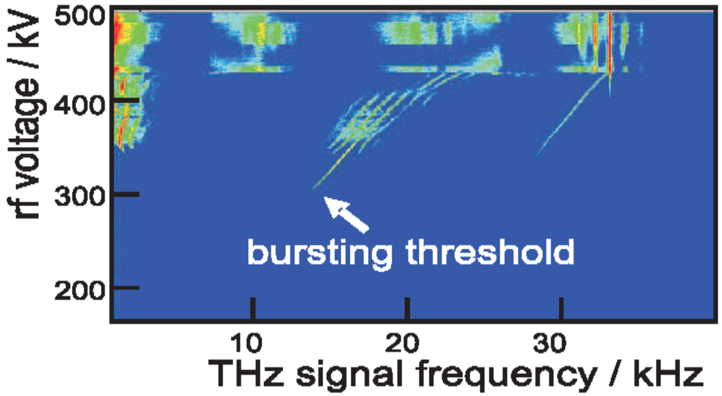


Figure 1: MLS THz signals at the bursting threshold. Vertical axis: applied rf-voltage amplitude, horizontal axis: frequency of the detected THz signals. The colour indicates the THz signal intensity.

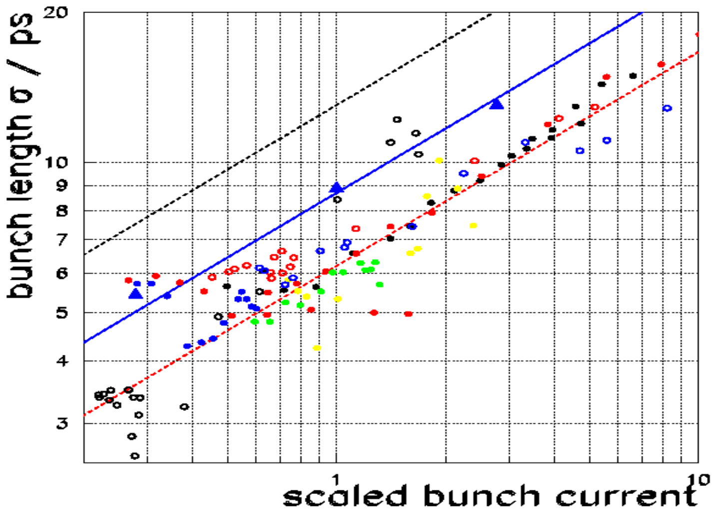


Figure 2: Scaled bursting threshold current $\tilde{I}$ and bunch length σ_0. Blue line: theory; blue triangles: BESSY II data; circles: MLS data. All data are measured with a liquid He cooled InSb detector of one MHz frequency response.

$$\sigma_z^{7/3} = \frac{c^2 Z_0}{2\pi F 3^{1/3}} I_b^{th} \rho^{1/3} / (V_{rf} f_{rf} f_{rev})$$

- This formula was derived from the coasting beam theory developed by Stupakov and Heifets. F = 7.456 is a form factor.
- According to Wustefeld, F=7.456, for BESSY II data and F= 3.4 for MLS data

Threshold of Instability
Comparing theory with measurement

A similar equation can be derived in the bunched beam theory (in MKS units)

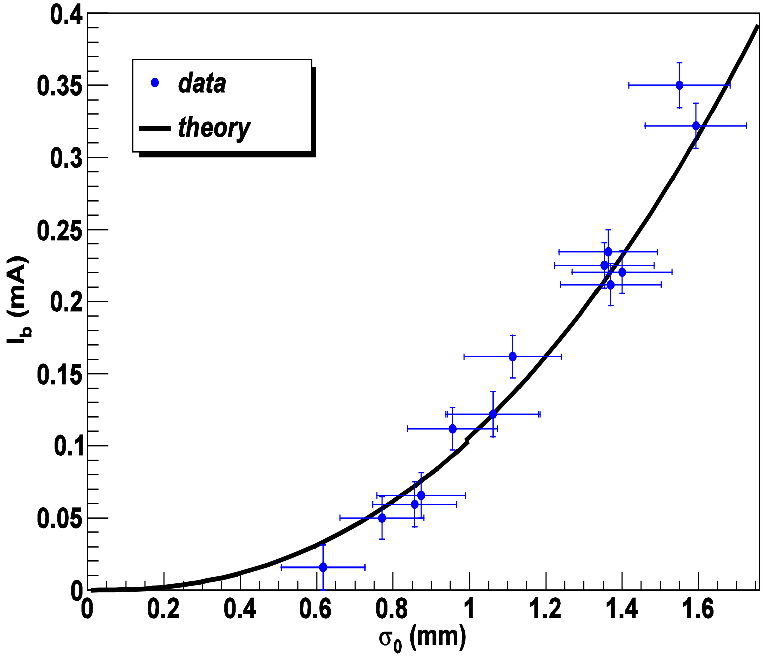
$$\sigma_z^{7/3} = \frac{c^2 Z_0}{8\pi^2 \xi^{th}(\chi)} I_b^{th} \rho^{1/3} / (V_{rf} \cos\varphi_s f_{rf} f_{rev})$$

It is nearly the same as the formula from the coasting beam theory, provided

$$F = 4\pi\xi^{th}(\chi)/3^{1/3}$$

But this theory gives a reason why F varies from machine to machine.

Measured bursting threshold at ANKA
See M.Klein et al. PAC09, 4761 (2009)



(courtesy of M. Klein, ξ^{th}=0.5 used.)

Summary of the Comparisons

Machine	σ_z [mm]	Radius ρ [m]	Gap h [cm]	χ	ξ^{th} (theory)	ξ^{th} (meas.)
BESSY II	2.6	4.23	5.0	0.48	0.67	0.89
MLS	2.6	1.53	5.0	0.29	0.60	0.39
ANKA	1.0	5.56	3.2	0.42	0.64	0.50
SSRL	1.0	8.14	3.4	0.46	0.66	?
Diamond	0.7	7.13	3.8	0.25	0.17 ?	0.33

We have used

$$\xi^{th}(\chi) = 0.5 + 0.34\chi$$

where $\chi = \sigma_z \rho^{1/2}/h^{3/2}$ is the shielding parameter. This simple relation was first obtained by fitting to the result of simulations (K. Bane, Y. Cai, and G. Stupakov, PRSTAB **13**, 104402 (2010)).

Since the MLS's shielding parameter is very close to the dip. That may be a reason of its lower threshold.

Conclusion

1) For a long bunch, $\chi = \sigma_z \rho^{1/2}/h^{3/2} > 2$, the coasting beam theory works well. The beam becomes unstable when

$$I_b > \frac{3\sqrt{2}\alpha\gamma\sigma_\delta^2 I_A \sigma_z}{\pi^{3/2} h}$$

2) When a bunch is short, $\chi < 2$, the bunched beam theory should be applied. The beam becomes unstable if

$$I_b > \frac{8\pi^2 \xi^{th}(\chi)\sigma_z^{7/3} V_{rf} \cos\varphi_s f_{rf} f_{rev}}{c^2 Z_0 \rho^{1/3}}$$

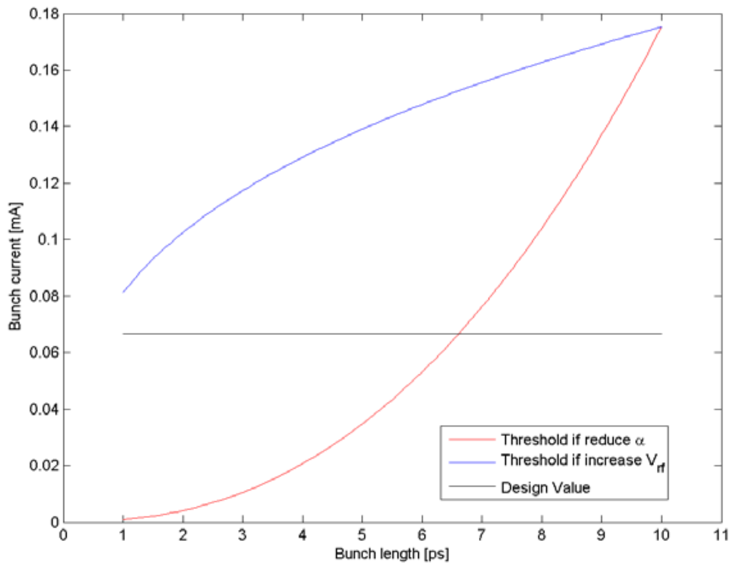
A shorter bunch is always more unstable. However, it is much better to reduce the bunch length with an increase of RF voltage than with a decrease of the momentum compaction factor.

3) Avoid the dip near $\chi = 0.25$.

Reduce Bunch Length from 10 ps to 1 ps without reducing bunch current

- For a high energy collider, it means a potential increase of luminosity by a factor of 10.
- For a synchrotron light source, this is a factor of 10 increase in time resolution. Also, a factor of 10 increase of peak current, may enable an XFEL oscillator.
- For a THz radiation source, this gives a potential gain of $10^4/4$, comparing to the low α approach.

Calculation of threshold



An illustration using PEP-X nominal parameters: f_{rf} = 476 MHz, V_{rf}=8.3 MV, f_{rev} = 136.312 kHz, σ_z=3 mm, I_b=0.067 mA.

Acknowledgements

- My colleagues: Karl Bane, Alex Chao, Gennady Stupakov, and Bob Warnock for many helpful and stimulating discussions, their insights, collaborations, and encouragements
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