The 6th Symp. on Accelerator Science and Technology, 1987, Tokyo, Japan (© Ionics, Tokyo, 1987)

RESONANCES IN CYCLOTRON
— EXISTENCE OF ν_r = 0/0 RESONANCE —

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ABSTRACT

An unbalance of acceleration voltages generates a motion of the orbit center. The radial betatron amplitude grows by a perturbation relating to energy gain at cavities and a radial periodic field distribution. The detailed study of this effect shows an existence of the radial betatron resonance ν_r = 0/0 for an accelerating case.

INTRODUCTION

Many trim coils are used to adjust the radial profile of the magnetic field in cyclotron. This corrected field is the sum of the isochronous field and a sinusoidal field. When a radial increase in $1/(\nu_r-1)$ revolutions coincides with the period of radial sinusoidal field, a radial resonance occurs.[1] The radial position of this resonance depends on the acceleration voltage and not the radial betatron frequency ν_r. This paper describes the resonance conditions which includes cavity voltages and a radial sinusoidal field component.

ACCELERATED ORBIT

A theory of accelerated orbit in the median plane of a cyclotron was developed related with the electric gap-crossing resonance.[2] This procedure can be used for gap-crossing and sinusoidal field.

The instantaneous equilibrium orbit

$$r_e(p,\theta) = R(1 + \xi(R,\theta)) \tag{1}$$

is chosen as a reference orbit, where R is the average radius and ξ has zero average value. An accelerated orbit is given by

$$R(\theta) = r_e(p,\theta) + x(\theta), \tag{2}$$

where $x(\theta)$ is the displacement of the orbit from the reference orbit, and assume that the amplitude of $x(\theta)$ is smaller than the period of radial periodic field component. The Lagrangian for the polar-coordinate trajectory $r = r(\theta)$ is expanded in powers of x and $\dot{x}$:

$$L(x,\dot{x},\theta) = (1/2)A(\theta)\dot{x}^2 + (1/2)B(\theta)x^2 + C(\theta)x + \Delta L. \tag{3}$$

In order to reduce the Lagrangian (3) to standard form, introduce a new coordinate $X(\theta)$ defined by:

$$x(\theta) = (A^{-1/2})X(\theta). \tag{4}$$

Then the Lagrangian (3) reduces to:

$$L(X,\dot{X},\theta) = (1/2)\dot{X}^2-(1/2)G(\theta)X^2+K(\theta)X+\Delta L. \tag{5}$$

The function $G(\theta)$ can be split into two parts:

$$G(\theta) = G_0(p,\theta) + \delta G(\theta), \tag{6}$$

where G_0 is the linear radial oscillation function about the equilibrium orbit of momentum p. For no acceleration, the function $G(\theta)$ reduces to G_0.

An independent variable ϕ and a coordinate $y = y(\phi)$ is introduced by

$$X(\theta) = W(\theta)y(\phi),$$

$$\phi = \theta + \psi,$$

$$W^2(1+\dot{\psi}) = 1. \tag{7}$$

The differential equation for $y(\phi)$ is

$$d^2y(\phi)/d\phi^2 + \nu_r^2 y(\phi) = F(y,\phi), \tag{8}$$

where $F(y,\phi)$ is the force associated with the magnetic field $B(r,\theta)$ and the momentum change at each gap. This force involves the quantity λ defined by

$$\lambda(\theta) = \dot{R}/R, \tag{9}$$

where dot denotes derivative with respect to θ. At each gap-crossing λ changes discontinuously. For the cyclotron with the gap-crossing number N_C and the same energy gain at each gap-crossing,

$$\lambda(\theta) = \lambda_0(1+2\sum_{n=1}^{\infty}\cos nN_C\theta). \tag{10}$$

RESONANCE CONDITIONS

(1) No Acceleration

In the absence of any acceleration, the function F has the form

$$F = y^{m-1}B_n , \tag{11}$$

where B_n is the n-th Fourier component of the magnetic field B. In this case the resonance condition is:

$$m\nu_r = n. \tag{12}$$

(2) Acceleration

In the case of acceleration, the function F has the form

$$F = \lambda_L y^{m-1}B_n , \tag{13}$$

where λ_L is the L-th Fourier component of the λ. In this case the resonance condition is:

$$m\nu_r = n \pm L. \tag{14}$$

(3) Acceleration and the Radial Periodic Field

The magnetic field of cyclotron is defined by

$$B(r,\theta) = B_0(r)(1+\sum_n b_n \cos n(\theta-\theta_n)) + B_T(r,\theta)$$

$$B_T(r,\theta) = b_t(r)(1+\sum_n B_{tn}(r,\theta))$$

$$b_t(r) = c\cos(2\pi r/d + \delta)$$

$$B_{tn}(r,\theta) = b_n(r)\cos n(\theta-\theta_n), \tag{15}$$

where $B_T(r,\theta)$ is the radial periodic field produced by trim coils. The parameters c and d are amplitude and period of radial field, respectively. When a radial increase of ions in $1/(\nu_r-1)$ revolutions coincides with the period of radial sinusoidal field d, the radial dependence of the field B_t can be expanded as

$$b_t(r) = \sum_{n=1}^{\infty} C_n \cos n((\nu_r-1)\theta+B_n). \quad (16)$$

Only n=1 component is large and the contribution of higher components is usually small.

In this case the function F has the form

$$F = \lambda_L y^{m-1} B_n b_k, \quad (17)$$

where b_k is the k-th Fourier component of the radial periodic field (16). In this case the resonance condition is:

$$(m \pm k)\nu_r = n \pm L \pm k \quad (18a),$$

with

$$2\pi R/d = (\nu_r - 1)/j. \quad (18b)$$

If the relation $M \pm k \neq 0$ is satisfied, then the resonance condition is Eqs. (18a) and (18b). Equation (18a) is a condition for the radial betatron frequency. Equation (18b) is a condition between the betatron frequency and a cavity voltage or an energy gain per turn. This condition seldom occurs in actual acceleration.

If both $m \pm k=0$ and $n \pm L \pm k=0$ are satisfied in Eq. (18a) and Eq. (18b) is also satisfied, there seems that $\nu_r = 0/0$ resonance exists. In this case the radial resonance is depend on not the radial betatron frequency ν_r but the energy gain condition (18b). The order of the resonance is zero, and it is a strong resonance. The resonance condition is not severe in this case, and the resonance always occurs at a radius where Eq. (18b) is satisfied.

Similar relation exists on the axial motion.

(4) Azimuthal Condition and Radial Condition

Equation (18a) and (18b) show that there exists not only an azimuthal condition but also a radial condition in determining the resonance of the radial motion of accelerating ions. In a well known case the radial periodic field does not exist, and the condition (18b) is not necessary. The resonance frequency ν_r is determined only from the azimuthal condition (18a) arizing from an azimuthal periodic variation of the magnetic field which is generated by hills and valleys of the cyclotron magnet or sector magnets and free spaces between magnets of the ring cyclotron.

When the equations $m \pm k=0$ and $n \pm L \pm k=0$ are satisfied, the azimuthal condition (18a) is helpless to dertermine the resonance frequency ν_r, and only the radial condition (18b) arises from a radial periodic variation of the magnetic field by introducing the correction field using trim coils.

PERTURBATION TERMS

(1)Radial Motion

A gap geometry is assumed as follows. There are N_C gap-crossings at $\theta = 2\pi/N_C$ and the same delta-function type energy gain at each gap-crossing, and a flat-topping gap is located at θ_f. The value of $\lambda(\theta)$ is expanded to

$$\lambda(\theta) = \lambda_0(1+2\sum_{n=1}^{\infty}\cos nN_c\theta)$$
$$+ \delta\, \lambda_0(1+2\sum_{n=1}^{\infty}\cos n(\theta-\theta_f)). \quad (19)$$

The perturbation term (17) relating to radial periodic field becomes

$$F(y,\phi) = -(pR)^{1/2}(k_0 + ((3/2)k_0 + k_1)\xi$$
$$+ 3k_0 w + (3/2)\lambda k_0 \xi_\theta)$$
$$-y(\phi)(k_0 + k_1 + (k_0 + 3k_1 + k_2)\xi$$
$$+ 3\lambda(k_0 + k_1)\xi_\theta), \quad (20)$$

where terms higher than first-order in ξ, w and λ and the second and higher order terms in $y(\theta)$ have been neglected. The quantities ξ_θ, ξ_r, w, k_0, k_1 and k_2 are defined by

$$\xi_\theta = \partial\xi/\partial\theta, \qquad \xi_r = R(\partial\xi/\partial R),$$
$$W(\theta) = 1 + w(\theta), \qquad k_0 = B_T/B_0,$$
$$K_1 = (R/B_0)(dB_T/dR) \text{ and } k_2 = (R^2/B_0)(d^2B_T/dR^2). \quad (21)$$

(2)Axial Motion

The differential equation for the axial motion $y_z(\phi)$ is

$$d^2y_z(\phi)/d\phi^2 + \nu_z^2 y_z(\phi) = F_z(y_z,\phi), \quad (22)$$

where $F_z(y_z,\phi)$ is the force associated with the magnetic field $B(r,\theta)$ and the momentum change at each gap. The general form for F_z is

$$F_z = \lambda_L y_z^{m-1} B_n b_k, \quad (23)$$

where b_k is the k-th Fourier component of the radial periodic field (16). In this case the resonance condition is:

$$(m \pm k)\ \nu_z = n \pm L \pm k\ , \quad (24a),$$

with

$$2\pi R/d = (\nu_z - 1)/j\ . \quad (24b)$$

The axial perturbation y_z relating radial periodic field becomes

$$F_z(y_z,\phi) =$$
$$y_z(\phi)(R(1 + 2\xi + \lambda\xi_\theta + 4w_z)(\partial B/\partial r)/B_0$$
$$-\{\xi_\theta + \lambda(1 + \xi + \xi_r + 4w)\}(\partial B/\partial\theta)/B_0). \quad (25)$$

The lowest term of F_z is a term with m=2 as seen from Eq. (25) and the contribution of higher terms with k>1 is expected to be small. Therefore the relation $m \pm k \neq 0$ is satisfied, and there is no $\nu_z = 0/0$ resonance.

PERTURBATION EFFECT

The rate of energy gain for the particle by delta-function type gap-crossing at $\theta=\theta_i$ is expressed as

$$\dot{E}_i(\theta) = (E_I/2\pi)(1 + 2\sum_{n=1}^{\infty}\cos n(\theta=\theta_i)). \quad (26)$$

The ion energy at an angle θ is evaluated by

$$E(\theta) = \int_0^\theta (\sum_{i=1}^{N}\dot{E}_i(\theta))d\theta$$
$$= E_I + (E_1/2\pi)\theta + E_1\sum_{n=1}^{\infty}(e_n/n\pi)\sin n(\theta-\theta_n'), \quad (27)$$

where N is the number of cavities, E_I is the initial energy and E_1 is the energy gain in one turn. The factor $(pR)^{1/2}$ in the perturbation $F(y,\phi)$ is given by

$$f(\phi) = (pR)^{1/2}$$
$$= f_F(\phi)(1+(E_1/2E_F(\phi))\sum_{n=1}^{\infty}(e_n/n\pi)\sin n(\phi-\phi_n)), \quad (28)$$

where

$$E_F(\phi) = E_I + (E_1/2\pi)\phi$$

and

$$f_F(\phi) \cong (2E_F(\phi)/eB_0)^{1/2}.$$

The particle energy $E_F(\phi)$ and $f_F(\phi)$ are monotonic functions of angle ϕ.

SINGLE FLAT-TOPPING GAP AND FIRST HARMONIC FIELD

In the case of N_c gap-crossings at the same angular interval and the same energy gain at each gap-crossing, $f(\theta)$ has only $n=mN_c$ Fourier components. However, in the case of single flat-topping gap $f(\theta)$ has all Fourier components on n values. The first term of $F(y,\phi)$ is reduced to

$$E_1(\phi) = -(pR)^{1/2}k_0$$
$$= -(pR)^{1/2}b_t(R)(1+b_1(R)\cos(\theta-\theta_1)+\cdots)/B_0(R). \quad (29)$$

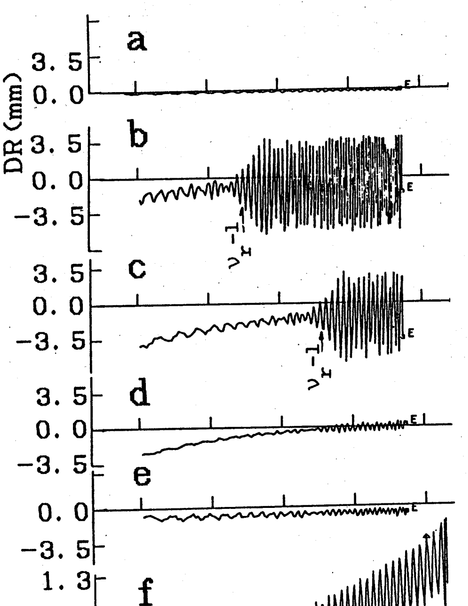

Fig. 1. The orbit motions of the 207 MeV protons in a six sector cyclotron with three dee cavities and a flat-topping cavity are shown by the radial displacement from the equilibrium orbit. The magnetic field is assumed to include radial sinusoidal field with amplitude 3 Gauss and period 4.5 cm.
V_i = the voltage of dee cavity No. i (i=1, 2 and 3)
V_f = the voltage of flat-topping cavity
a) $V_1=V_2=V_3=125$ kV, $V_f=0$ kV. There is no resonance.
b) $V_1=V_2=V_3=125$ kV, $V_f=-66.2$ kV. The resonance appears by a flat-topping gap crossing.
c) $V_1=V_2=V_3=250$ kV, $V_f=-132.5$ kV. The resonance shifts to larger radius.
d) $V_1=190$ kV, $V_2=V_3=280$ kV, $V_f=-132.5$ kV. the resonance effect is cancelled by a rearrangemnt of cavity voltages.
e) $V_1=V_2=V_3=250$ kV, $V_f=-132.5$ kV. The resonance effect is cancelled by the first harmonic field. The magnetic fields of two sector magnets that confronts each other are changed ±0.025%.
f) The oscillation of radial betatron frequency ν_r shows the periodicity of radial sinusoidal field.

Now consider the case that the radial increase of ion orbits in $1/(\nu_r-1)$ revolutions is equal to the period of radial sinusoidal field. Since b_t has strong $n=\nu_r-1$ Fourier component and $(pR)^{1/2}$ has predominantly $n=1$ component for single flat-topping cavity, then $F_1(\phi)$ will have an $n=\nu_r$ component. If the first harmonic component is present in the magnetic field, from this component and strong $n=\nu_r-1$ component of radial sinusoidal field the perturbation $F_1(\phi)$ will have an $n=\nu_r$ component. The force in the presence of the flat-topping cavity is proportional to $E_1/2E_F$, where E_1/E_F is the ratio of the energy gain per turn to the ion energy. The force in the presence of the first harmonic field is proportional to $b_1(R)$ in cyclotron unit, i.e., the value $b_1(R)$ is the ratio of the strength of the first harmonic field to the average field. Since the other factors are common to two forces, it is possible to cancel these forces by selecting the voltage of flat-topping cavity and the strength and phase of the first harmonic field. Another method to suppress the force $F_1(\phi)$ is to eliminate the periodic parts of $f(\phi)$ by choosing suitable voltages of cavities. The computer results on the six sector cyclotron are shown in Fig. 1. The amplitude growth by this resonance is 5 mm to 10 mm, and this is compatible to the turn separations at larger radius. Therefore it is necessary to eliminate this effect. The figures also show that such cancellation is possible.

GAP-CROSSING RESONANCE

The perturbation $F(\phi)$ has gap-crossing resonance terms:

$$F_2(\phi) = -(pR)^{1/2}(3/2)\lambda k_0\xi_\theta$$
$$-3\lambda(k_0 + k_1)\xi_\theta y(\phi). \quad (30)$$

In the case of gap-crossing resonance[2] the perturbation has a term

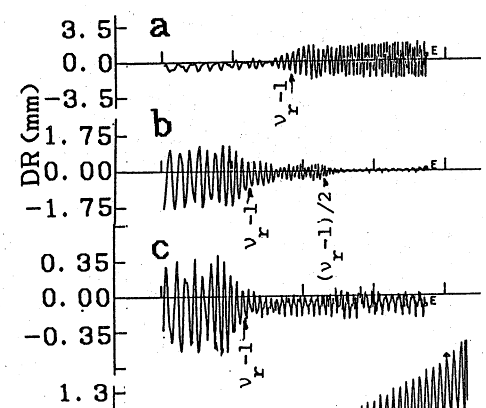

Fig. 2. The orbit motions of the 207 MeV protons in a six sector cyclotron with five and seven cavities are shown by the radial displacement from the equilibrium orbit. The magnetic field is assumed to include radial sinusoidal field with amplitude 3 Gauss and period 4.5 cm.
a) five cavities, V=125 kV. The gap-crossing resonance.
b) five cavities, V=90 kV. The proton is decelerated from larger radius. There are two gap-crossing resonances.
c) seven cavities, V=60 kV. The proton is decelerated from larger radius. The effect of gap-crossing resonance is small.

$$F_G(\phi) = -(pR)^{1/2}\lambda(2+k)\xi_\theta , \quad (31)$$

where k is the field index, but it does not have a radial periodic field. Even if a radial periodic field is added in the perturbation, similar resonance condition as that of the gap-crossing resonance holds.

To evaluate the effect of this resonance, consider an unreal six sector cyclotron with five or seven gap-crossings at the same angular interval and the same energy gain at each gap-crossing. The field distribution for 207 MeV proton is used in the calculation. The value λ has strong n=5 and n=7 Fourier components for five and seven gap-crossings, respectively. Moreover, since ξ_θ has predominantly an n=6 component and the radial field k_0 has strong $n=\nu_r-1$ component, then the first term of $F_2(\phi)$ will have an n=1 component.

The radial field index k_0 also has a weak $n=2(\nu_r-1)$ Fourier component. The value λ has n=10 and n=14 Fourier components for five and seven gap-crossings, respectively. Since ξ_θ has an n=12 component and function $y(\phi)$ has $n=\nu_r$ component, then the second term of $F_2(\phi)$ will have an n=1 component.

Fig. 2 shows some computer results of the gap-crossing resonance including a radial periodic field. There are two resonances during ion acceleration. The resonances occur at radii where the radial increases of ion orbits in $1/(\nu_r-1)$ and $2/(\nu_r-1)$ revolutions are equal to the period of radial sinusoidal field. Similar gap-crossing is also observed in computer calculations on four sector cyclotron with three or five cavities as shown in Fig. 3.

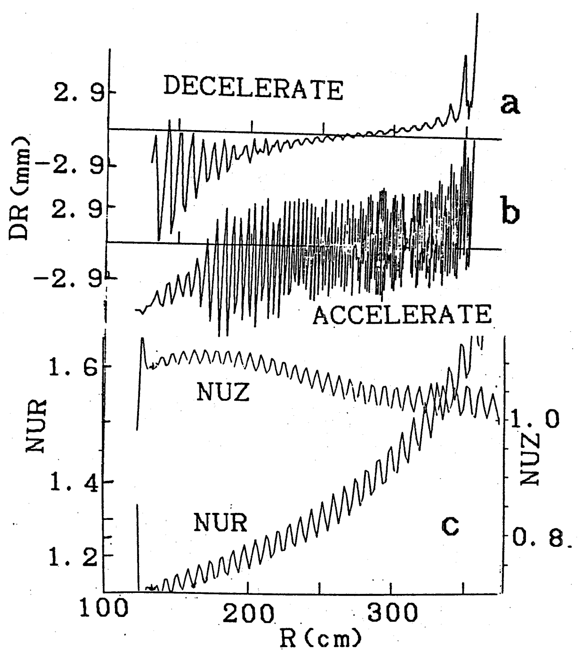

Fig. 3. The orbit motions of the 300 MeV protons in a four sector cyclotron with three cavities are shown by the radial displacement from the equilibrium orbit. The cavity voltages are V=600 kV. The magnetic field is assumed to include radial sinusoidal field with amplitude 3 Gauss and period 8.5 cm.

a) The proton is decelerated from larger radius. The gap-crossing resonance.

b) The proton is accelerated from smaller radius. The gap-crossing resonance.

c) The oscillations of betatron frequencies ν_r and ν_z show the periodicity of radial sinusoidal field. The resonance occurs at radius where the period of radial field (Fig. c) is equal to that of betatron oscillation of the ion orbit (Figs. a and b).

Since in the gap-crossing of the flat-topping cavity λ also has not only n=1 Fourier component but also n=5 and n=7 components, both the first and the second terms of the perturbation $F_2(\phi)$ have an n=1 component in a cyclotron with single flat-topping cavity.

For the six sector cyclotron the force by a single flat-topping gap (energy gain by this gap/turn = -56 kV) can be compared with the force by gap-crossing resonance of five accelerating gaps (energy gain by five gaps/turn = 450 kV). It is assumed that the ion energy is 100 MeV, and the amplitude of the equilibrium orbit ξ is 0.04. The force by a flat-topping gap is propotional to $E_1/2E_F = 0.00028$. Since the amplitude of ξ is 0.04, the amplitude of ξ_θ is 0.04xN, where N is the number of sectors. The amplitude of N-th component in λ is evaluated to

$$\lambda_N = E_N/(2\pi E_F(1 + k)) = 0.00072 \quad (32)$$

for five accelerating gaps, where k is the field index, and E_N is the energy gain by N gaps per turn. The $\lambda_5\xi_\theta$ value for five accelerating gaps is $\lambda_5\xi_\theta = 0.00014$. By using these values the ratio of the force by a flat-topping gap to that by five accelerating gaps becomes to 2, and is equal to the ratio of amplitude growths of radial betatron oscillation. The amplitude growth by the gap-crossing resonance in six sector cyclotron with seven cavities is very small, and is not interpreted by the force $F_2(\phi)$.

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