Focal Characteristic of Helical Quadrupole Focusing Channel

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Abstract

A helical quadrupole focusing channel has continuous field symmetry and stronger focusing power compared with a conventional FODO focusing channel. The good field symmetry allowed us to construct the explicit transfer matrix under the paraxial approximation. The paraxial analysis of the helical quadrupole focusing channel and the characteristic of the point-to-point focusing are reported.

1 INTRODUCTION

A helical quadrupole focusing channel(HQFC) is a variation of a quadrupole focusing channel, which consists of a long quadrupole field continuously twisted along the beam axis. Its strong focus is adequate for a transport of a large emittance beam such as secondary particles. Because focal strength of HQFC, unlike that of a solenoid channel, depends on total momentum of a particle, it can be used as a momentum filter for a beam from a point-like source. In the conventional analysis, the HQFC has been treated as a series of rotating quadrupole blocks[1]. Accurate analysis, however, requires a treatment in a magnetic field distribution that satisfies the Maxwell equation. The analysis and fabrication of a electrostatic helical quadrupole were reported in Ref. [2, 3, 4]. In the subsequent section, the field distribution of the HQFC, the analytic solution of the transfer matrix under the paraxial approximation and one of the focal characteristic are presented. The comparison with a FODO and a solenoid is described in Ref. [5].

2 HELICAL QUADRUPOLE FIELD

In a free space, the helical quadrupole field satisfying the Maxwell equation is expressed by the following form using modified Bessel function

$$\Phi(r,\theta,z) = \frac{\Phi_0}{k} I_2(kr)\sin(2\theta - kz) \quad (1)$$
$$\text{and} \quad \mathbf{B} = -\nabla\Phi, \quad (2)$$

where k is $2\pi/L$ and L is the period of the magnetic field (the period of a pole is $2L$). Figure 1 shows one of the pole surface of the helical quadrupole that is a equal potential surface of Eq. (1). The field component in cylindrical coordinate system is given as follows:

$$B_r(r,\theta,z) = -\Phi_0 I_2'(kr)\sin(2\theta - kz), \quad (3)$$
$$B_\theta(r,\theta,z) = -2\Phi_0 \frac{I_2(kr)}{kr}\cos(2\theta - kz) \quad (4)$$
$$\text{and} \quad B_z(r,\theta,z) = \Phi_0 I_2(kr)\cos(2\theta - kz). \quad (5)$$

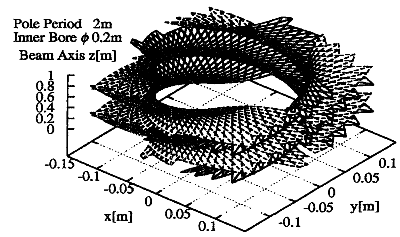

Figure 1: The pole surface of the helical quadrupole

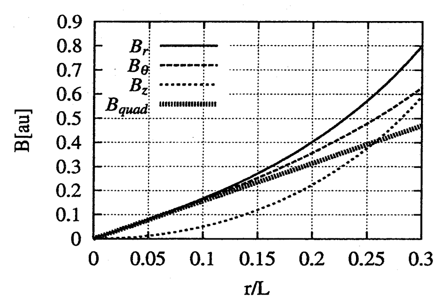

Figure 2: r dependence of radial part of both helical and pure quadrupole fields: B_r,B_θ,B_z and B_{quad}

Figure 2 shows r dependences of the three components of the helical quadrupole field together with B_r component of the pure quadrupole. Around the central axis, the helical quadrupole field can be approximated by a rotating quadrupole with the angle of $\frac{kz}{2}$ radian. This paraxial approximation corresponds to the linear approximation. In order to examine the valid range of the linear approximation, we obtain following expansions by Taylor series:

$$B_r = -\frac{\Phi_0}{4}(kr + \frac{(kr)^3}{6} + O(r^5))\sin(2\theta - kz), \quad (6)$$
$$B_\theta = -\frac{\Phi_0}{4}(kr + \frac{(kr)^3}{12} + O(r^5))\cos(2\theta - kz) \quad (7)$$
$$\text{and } B_z = \frac{\Phi_0}{8}((kr)^2 + \frac{(kr)^4}{12} + O(r^6))\cos(2\theta - kz). \quad (8)$$

For a good linear approximation, the nonlinear terms in B_r, B_θ and B_z components have to be negligible compared with the pure quadrupole component. If 10% deviation is allowed for either B_r or B_θ component, the maximum r coordinate r_{bore} has to keep following condition:

$$r_{bore} \lesssim 0.12L. \quad (9)$$

Same limitation for B_z component is given by

$$r_{bore} \lesssim 0.032L. \quad (10)$$

3 TRANSFER MATRIX OF HELICAL QUADRUPOLE FOCUSING CHANNEL

Linearized transfer matrix of the HQFC will be derived in this chapter. The stability condition for the matrix is deduced by Eigenvalue analysis.

3.1 Cell Matrix

Considering the rotating characteristics of the HQFC's magnetic scalar potential, a transfer matrix from location s_a to s_b $M_{HQ}(s_b|s_a)$ can be written as

$$M_{HQ}(s_b|s_a) = R(\frac{ks_a}{2})M_{HQ}(s_b - s_a|0)R(-\frac{ks_a}{2}), \quad (11)$$

where matrix R is a rotation matrix. Using the translation operator Eq. (11), a transfer matrix of the HQFC $M_{HQ}(s_{out}|s_{in})$ can be divided into products of small cell matrices:

$$\begin{aligned} M_{HQ}(s_{out}|s_{in}) &= \prod_{i=0}^{N-1} M_{HQ}(s_{i+1}|s_i) \\ &= \prod_{i=0}^{N-1} R(\frac{ks_i}{2})M_{HQ}(s_{i+1} - s_i|0)R(-\frac{ks_i}{2}) \\ &= R(\frac{ks_{out}}{2})\left(\prod_{i=0}^{N-1} R(-\frac{k(s_{i+1}-s_i)}{2})\right. \\ &\qquad \left. M_{HQ}(s_{i+1} - s_i|0)\right) R(-\frac{ks_{in}}{2}), \quad (12) \end{aligned}$$

where $\{s_i | i = 0, \ldots, N, s_0 = s_{in}, s_N = s_{out}\}$ is a division of the region $[s_{in}, s_{out}]$. Applying the equal weight division and a limit operator to Eq. (12), we obtain

$$\begin{aligned} M_{HQ}(s_{out}|s_{in}) &= R(\frac{ks_{out}}{2}) \\ \lim_{N\to\infty} &\left(\prod_{i=0}^{N-1} R(-\frac{k\triangle s}{2})M_{HQ}(\triangle s|0)\right) R(-\frac{ks_{in}}{2}), \quad (13) \end{aligned}$$

where $\triangle s$ is the length of slice: $\triangle s = (s_{out} - s_{in})/N$. Considering that a thin helical quadrupole is equivalent to a thin quadrupole under the linear approximation, the transfer matrix $M_{HQ}(\triangle s|0)$ in Eq. (13) can be replaced by a transfer matrix of a thin quadrupole $M_Q(\triangle s|0)$. Let us introduce the matrix M_{core} to denote the infinity matrix product in Eq. (13) as follows:

$$\begin{aligned} M_{core}(s) &= \lim_{N\to\infty}\left(\prod_{i=0}^{N-1} R(-\frac{ks}{2N})M_{HQ}(\frac{s}{N}|0)\right) \\ &= \lim_{N\to\infty}\left(\prod_{i=0}^{N-1} R(-\frac{ks}{2N})M_Q(\frac{s}{N}|0)\right). \quad (14) \end{aligned}$$

The infinite matrix product M_{core} is the products of the cell matrix $R(-ks/2N)M_Q(s/N|0)$. The explicit representations of $R(d\theta)$ and $M_Q(ds|0)$ in the transversal phase space(x, x', y, y') are given by

$$R(d\theta) = \begin{pmatrix} \cos d\theta & 0 & -\sin d\theta & 0 \\ 0 & \cos d\theta & 0 & -\sin d\theta \\ \sin d\theta & 0 & \cos d\theta & 0 \\ 0 & \sin d\theta & 0 & \cos d\theta \end{pmatrix} \quad (15)$$

$$\text{and } M_Q(ds|0) = \begin{pmatrix} 1 & ds & 0 & 0 \\ -Kds & 1 & 0 & 0 \\ 0 & 0 & 1 & ds \\ 0 & 0 & Kds & 1 \end{pmatrix}, \quad (16)$$

where K, q and p are the field gradient of the quadrupole field $(-qk\,\Phi_0/4p)$, the charge and the momentum of the particle, respectively. Because the sign of the parameter K can be chosen without loss of generality, we will assume that K is positive hereafter.

3.2 Eigenvalue Analysis and Stability Condition

Let us derive the Eigenvalue and the Eigenvectors of the infinity product in Eq. (14) from those of the cell matrix. The Eigenvalue problem of the cell matrix indexed by s/N is defined as

$$\hat{\lambda}_i(s/N)\hat{\mathbf{x}}_i(s/N) = R(-\frac{ks}{2N})M_Q(s/N|0)\hat{\mathbf{x}}_i(s/N). \quad (17)$$

Using the Eigensolutions of Eq. (17), the Eigesolutions of the infinity product M_{core} are described as follows:

$$\mathbf{x}_i = \lim_{N\to\infty} \hat{\mathbf{x}}_i(s/N), \quad \lambda_i = \lim_{N\to\infty} \hat{\lambda}_i(s/N)^N. \quad (18)$$

After some mathematical operations, we can obtain the Eigensolution of the matrix M_{core} as follows:

$$(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = \left(e^{-i\omega_H s}, e^{+i\omega_H s}, e^{-i\omega_L s}, e^{+i\omega_L s}\right), \quad (19)$$

where ω_H and ω_L are $\sqrt{k^2/4 + K}$ and $\sqrt{k^2/4 - K}$, respectively. The set of the Eigenvectors is

$$X = [\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4] = \begin{pmatrix} i\frac{2\omega_H}{k} & -i\frac{2\omega_H}{k} & \frac{k}{2K} & \frac{k}{2K} \\ \frac{2K}{k} & \frac{2K}{k} & 0 & 0 \\ 1 & 1 & -i\frac{\omega_L}{K} & i\frac{\omega_L}{K} \\ 0 & 0 & 1 & 1 \end{pmatrix}, \quad (20)$$

where the vector set X forms the basis of the Eigenspace. When both ω_H and ω_L are real, the solutions are periodic and the corresponding motion of a particle is stabilized. Thus the stability condition is given by

$$0 \le K \le \frac{k^2}{4} = \frac{\pi^2}{L^2}. \quad (21)$$

The transfer matrix M_{HQ} can be constructed by λ_i and X as follows:

$$M_{HQ}(s_{out}|s_{in}) = R(\frac{ks_{out}}{2})M_{core}(s_{out} - s_{in})R(-\frac{ks_{in}}{2}),$$

$$M_{core}(s) = X \begin{pmatrix} \lambda_1 & & & \mathbf{0} \\ & \lambda_2 & & \\ & & \lambda_3 & \\ \mathbf{0} & & & \lambda_4 \end{pmatrix} X^{-1}. \quad (22)$$

Explicit representation of $M_{core}(s)$ is given by

$$M_{core}(s) = \begin{pmatrix} C_p & \frac{L}{g\pi}\left(g_p S_p - \frac{S_m}{g_m}\right) & \frac{S_m}{g_m} & \frac{L}{g\pi}\left(C_m - C_p\right) \\ -\frac{\pi}{L}\frac{g}{g_p}S_p & C_p & 0 & \frac{S_p}{g_p} \\ -\frac{S_p}{g_p} & \frac{L}{g\pi}\left(C_p - C_m\right) & C_m & \frac{L}{g\pi}\left(\frac{S_p}{g_p} - g_m S_m\right) \\ 0 & -\frac{S_m}{g_m} & \frac{\pi}{L}\frac{g}{g_m}S_m & C_m \end{pmatrix}, \quad \begin{array}{rcl} C_p & = & \cos g_p\bar{s} \\ C_m & = & \cos g_m\bar{s} \\ S_p & = & \sin g_p\bar{s} \\ S_m & = & \sin g_m\bar{s} \end{array}$$

$$\bar{s} = \frac{\pi s}{L},\ g_p = \sqrt{1+g},\ g_m = \sqrt{1-g} \text{ and } \quad g = K\left(\frac{L}{\pi}\right)^2. \qquad (23)$$

4 POINT-TO-POINT FOCUSING

Considering the transportation of a secondary beam, the point-to-point focusing characteristic is useful. Let us discuss about the condition of the point-to-point focusing.

The transfer matrix in Eq. (22) is described by the matrix M_{core} except a coordinate rotation. These coordinate rotation matrix is unrelated to the property of the point focusing. The core matrix M_{core} has two different frequency of the oscillation in the Eigenspace and these two oscillation is not isolated in the real phase space. Thus in order to focus by point-to-point, two oscillation phases $\omega_H s$ and $\omega_L s$ have to be multiples of π and the difference of two phases $(\omega_H - \omega_L)s$ should be a multiple of 2π:

$$\exists s, p, q, n \quad \omega_H s = p\pi, \omega_L s = q\pi, (\omega_H - \omega_L)s = 2\pi n$$
$$s \in \mathbf{R}, \quad p, q, n \in \mathbf{N} \quad (24)$$

In order to obtain the minimum period of the point-to-point focusing, the rational number p/q should be irreducible rational. Thus the dimensionless focusing parameter G and the minimum period s_0 is obtained as:

$$\frac{G}{\pi^2} = \frac{4K}{k^2} = \frac{p^2 - q^2}{p^2 + q^2}, \quad s_0 = L\sqrt{\frac{p^2+q^2}{2}}$$
$$gcd(p,q) = 1, \quad p > q, \quad p, q \in \mathbf{Odd} \quad (25)$$

For example, the minimum period s_0 and the focusing parameter G, which corresponds to the minimum period s_0, is $0.8\pi^2 \sim 7.9$ and $\sqrt{5}L$, respectively.

Let us estimate the cross section of the beam from a point source. Using the notations in the transfer matrices in Eqs. (22),(23), a beam boundary that began from a point source is given by:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{Lr'}{g\pi}\begin{pmatrix} g_p S_p - \frac{S_m}{g_m} & C_m - C_p \\ C_p - C_m & \frac{S_p}{g_p} - g_m S_m \end{pmatrix}\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \qquad (26)$$

where r' and θ are the divergence of the initial beam and the azimuthal angle of initial beam, respectively. From the estimation of the capture range against the initial divergence under the linear approximation [5], the upper limit of the product Lr' is about 0.8 times of the bore radius of the HQFC.

Because the transformation in Eq. (26) is linear, the scale factor of the cross section is obtained as the determinant of the transformation matrix from the theory of the linear algebra. Thus the cross section $S(s,G)$ is written down as:

$$\frac{S(s,G)}{L^2 r'^2} = \frac{2\pi^2}{G^2}\left| 1 - \cos\frac{\sqrt{\pi^2 - G}s}{L}\cos\frac{\sqrt{\pi^2 + G}s}{L} \right.$$
$$\left. -\frac{2\pi^4 - G^2}{2\pi^2\sqrt{\pi^4 - G^2}}\sin\frac{\sqrt{\pi^2 - G}s}{L}\sin\frac{\sqrt{\pi^2 + G}s}{L})\right|. \quad (27)$$

The peaks of the logarithm of the $S(s,G)^{-1}$ in Eq. (27)

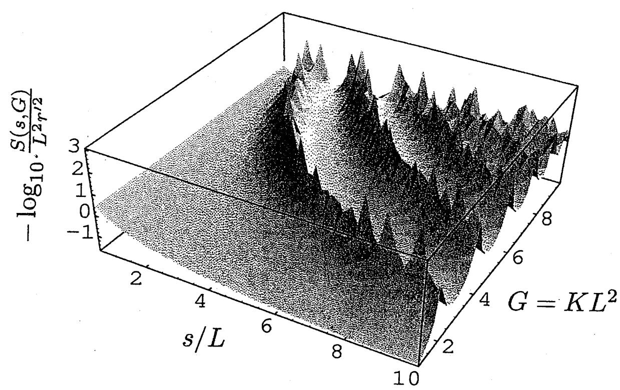

Figure 3: The cross section of the point source beam

shown in Fig. 3 are the points on which the beam cross section shrinks. Thus the HQFC might be usable as a momentum discriminator by combining with a collimator.

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